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t-BASICITY OF ROOT FUNCTIONS OF ONE NONLOCAL SPECTRAL PROBLEM FOR BOCHNER SPACE

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Abstract. We consider the system of functions, which are the root functions of one nonlocal spectral problem. It is introduced the concept of t-basis, induced by some tensor product. It is established the t-basisity of root functions for Bochner space $L_p(\mathcal{J};X)$, $1 < p < +\infty$, if X is UMD (Unconditional Martingale Difference) space, where $\mathcal{J} = (0,2\pi)$. For this goal we firstly define the concept of t-Riesz property for classical trigonometric system and prove that if X is UMD space, then the trigonometric system has t-Riesz property in $L_p(\mathcal{J};X)$, $1 < p < +\infty$.

Keywords: nonlocal spectral problem, t-basicity, t-Riesz property, Bochner space, Unconditional Martingale Difference property

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1. Introduction. Banach valued differential equations have applications in various areas of mathematics such as partial differential equations, stochastic evolution equations, harmonic analysis, spectral theory of linear (in particular of differential) operators and etc. By this reason the interest to the X-valued differential equations very hight. This theory in one variable case sufficiently well developed (see, e.g. the monographs [1-7]). In connection with the development of analysis in Banach spaces it began to be studied solvability questions of X-valued differential equations in many variable case (see, e.g. the works [8-15, 19- 21]).

We consider the system of functions, which are the root functions of one nonlocal spectral problem. It is introduced the concept of t-basis, induced by some tensor product. It is established the t-basicity of root functions for Bochner space $L_p(\mathcal{J};X)$, $1 < p < +\infty$, if X is UMD (Unconditional Martingale Difference) space, where $\mathcal{J} = (0,2\pi)$. For this goal we firstly define the concept of t-Riesz property for classical trigonometric system and prove that if X is UMD space, then the trigonometric system has t-Riesz property in $L_p(\mathcal{J};X)$, $1 < p < +\infty$.

Note that UMD property for B-space X provides boundedness for many operators of harmonic analysis in Bochner spaces. And this in turn allows us to generalize the concept of a basis in the context of a Banach tensor product and to prove that the classical exponent system and their generalization Faber polynomials form basis for corresponding Bochner spaces in this sense. Using these results in works [16-18] it is given the X-valued analogues of Hardy and Smirnov classes of analytic functions and it is established some basic properties of these classes.

2. Preliminaries.

2.1. Notations. Accept the following notations. $\mathbb{N}$ – natural numbers; $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$; $\mathbb{R}$ – real numbers; $\mathbb{C}$ – complex numbers; $\mathcal{J} = (0,2\pi)$; $I = (-\pi,\pi)$. $L[M]$ is a linear span of M; $\overline{M}$ is a closure of M; B-space will be a Banach space. $\|\cdot\|_X$ will stand for the norm in X. By $[X;Y]$ we will denote a B-space of bounded (linear) operators from X to Y with $[X] = [X;X]$.

2.2. t-basis properties. t-basis from exponents. Let X, Y, Z be B-spaces and $t: X \times Y \to Z$ be a bilinear operator satisfying the following condition $\exists \delta > 0: \delta \parallel x \parallel_X \parallel y \parallel_Y \leq \parallel t(x;y) \parallel_Z \leq \delta^{-1} \parallel x \parallel_X \parallel y \parallel_Y$, $\forall (x;y) \in X \times Y$.

For simplicity, future presentation accepts the notation $xy := t(x;y)$ for every $(x;y) \in X \times Y$.

We denote t-span of M by $L_t[M]$ for the set $M \subset Y$ and define it as

$$L_t[M] = \{z \in Z: \exists \{(x_k;y_k)\}_1^{n_0} \subset X \times M \Rightarrow z = \textstyle\sum_{k=1}^{n_0} x_k y_k\}.$$

Let $\tilde{y} \equiv \{y_k\}_{k\in\mathbb{N}} \subset Y$ be some system. Accept the following concepts.

System $\tilde{y}$ is t-complete in Z, if $\overline{L_t[\tilde{y}]} = Z$ (closure is taken in Z).

The system of operators $\{t_n\}_{n\in\mathbb{N}} \subset [Z;X]$ is called t-biorthogonal to $\tilde{y} \subset Y$, if $t_n(xy_k) = x\delta_{nk}$, $\forall x \in X \forall n, k \in \mathbb{N}$.

The system $\tilde{y} \subset Y$ forms t-basis for Z if $\forall z \in Z$ has a unique expansion in the form

$$z = \sum_{k=1}^{\infty} x_k y_k,$$

with $\{x_k\}_{k\in\mathbb{N}} \subset X$.

We call a triple $(X;Y;Z)$ be t_Y-invariant if $\{(x_k;\tilde{y}_k)\} \subset X \times Y: \sum_k x_k \tilde{y}_k = 0 \Rightarrow \sum_k \vartheta(\tilde{y}_k) x_k = 0$, $\forall \vartheta \in Y^*$.

Triple (X,Y,Z) is t-dense if $\overline{L[X \times Y]} = Z$ (closure is taken in Z).

The following criterion for t-basicity is valid.

Theorem 2.1 *Let the triple $(X;Y;Z)$ be t_Y-invariant and t-dense. Then the system $\tilde{y}$ forms a t-basis for Z if and only if the following assertions hold:*

- *(i) $\tilde{y}$ is t-complete in Z;*
- *(ii) $\tilde{y}$ has t-biorthogonal system $\{t_n\}_{n\in\mathbb{N}} \subset [Z;X]$;*
- *(iii) the projectors $\{P_m\}_{m\in\mathbb{N}}$:*

$$P_m(z) = \sum_{n=1}^{m} t_n(z) y_n, \qquad \forall z \in Z \forall m \in \mathbb{N},$$

are uniformly bounded, i.e. $\sup_m \parallel P_m \parallel_{[Z]} < \infty$.

We consider Z as the some Banach tensor product $X \overline{\otimes} Y$ of B-spaces X and Y. Denote the algebraic tensor product of XY by $X \otimes Y$ and the elementary tensor product of elements $x \in Xy \in Y$ by $x \otimes y$. In this case, it is obvious that the triple $(X;Y;Z)$ is $t -$ dense and $t_Y -$ invariant regarding the bilinear map $t(x,y) = x \otimes y$. Thus, according to the Theorem 2.1 we have the following

Corollary 2.2 *Let $X;Y$ be B-spaces and $Z = X \overline{\otimes} Y$. Then the system $\tilde{y} \subset Y$ forms $t -$ basis for Z if and only if the assertions $(i) - (iii)$ of Theorem 2.1 hold.*

2.3. Bochner Spaces and UMD Space. Let $(\mathcal{S},\mathcal{A},\mu)$ be a measure space and X be B-space. As usual, denote by $L_p(\mathcal{S};X)$, $1\le p<+\infty$, the Bochner space generated by measure space $(\mathcal{S};\mathcal{A};\mu)$ with norm

$$\| f \|_{L_p(\mathcal{S};X)} = \left(\int_{\mathcal{S}} \| f \|_X^p \, d\mu\right)^{\frac{1}{p}}.$$

The Bochner space $L_p(\gamma;X)$ is defined similarly. We identify the segment I_0 and unit circle γ by mapping $e^{it}: I_0 \to \gamma$. This allows us to identify also the spaces $L_p(I_0;X)$ and $L_p(\gamma;X)$.
We provide the definition of the UMD property and the associated space.
Definition 2.3 *A Banach space X is said to have the property of UMD, if for all $p\in(1,\infty)$ there exists a finite constant $\beta\ge 0$ (depending on p and X) such that the following holds: whenever $(\mathcal{S};\mathcal{A};\mu)$ is a $\sigma-finite$ measure space, $\{\mathcal{F}_n\}_{n=0}^N$ is a $\sigma-finite$ filtration and $\{f_n\}_{n=0}^N$ is a finite martingale in $L_p(\mathcal{S};X)$, then for all scalar $|\varepsilon_n|=1$, $n=\overline{1,N}$; we have*

$$\left\| \sum_{n=1}^N \varepsilon_n df_n \right\|_{L_p(\mathcal{S};X)} \le \beta \left\| \sum_{n=1}^N df_n \right\|_{L_p(\mathcal{S};X)}.$$

where $df_n = f_n - f_{n-1}$ is a martingale difference.
Let the set of all B-spaces that possess the UMD property be denoted by the symbol UMD.
To establish an analogous of the classical Fatou's theorem regarding harmonic functions on ω, we will need the following lemma from the monograph [21] (see p.127, Lemma 2.5.8).
Lemma 2.4 *[21] Let $g\in L_1^{loc}(\mathbb{R};X)$ and $a\in\mathbb{R}$ and define $f:\mathbb{R}\to X$ by*

$$f(t) =: \int_a^t g(s)ds$$

Then the weak derivative ∂f and almost everywhere derivative f' of f both exist in $L_1^{loc}(\mathbb{R};X)$ and are given by the $\partial f = f' = g$.
The set of all X-valued trigonometric polynomials $P_n: I_0 \to X$ of the form

$$P_n(t) = \sum_{k=-n}^{n} a_k e^{ikt},$$

with coefficients $\{a_k\}\subset X$, denote by $\mathcal{P}(X)$.
The following proposition is valid.
Proposition 2.5 *Let X be a B-space. Then $\overline{\mathcal{P}(X)} = L_p(I_0;X)$,$1\le p<\infty$) (closure is taken in $L_p(I_0;X)$).*
We define on $\mathcal{P}(X)$ the multiplier operator $m:\mathcal{P}(X)\to L_p(I_0;X)$ by expression

$$(mP)(t) = \tilde{P}(t) = -i\sum_{k\in\mathbb{Z}} sign(k) a_k e^{ikt},$$

where

$$P(t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt} \in \mathcal{P}(X),$$

and

$$sign(k) = \begin{cases} 1, & \text{if} k > 0, \\ 0, & \text{if} k = 0, \\ -1, & \text{if} k < 0. \end{cases}$$

We also consider the subspace $L_p^0(I_0;X)$ of $L_p(I_0;X)$ defined by

$$L_p^0(I_0;X) = \{f \in L_p(I_0;X): \int_{I_0} f(t)dt = 0\}.$$

Let H be the X-valued Hilbert transform on $\mathbb{R}$:

$$(Hf)(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy, x \in \mathbb{R},$$

defined in a singular sense. The following H-characterization of UMD property is known.

Theorem 2.6 *[Burkholder-Bourgain] Let X be a B-space $p \in (1,\infty)$. The following assertions are equivalent:*

- $X \in$ UMD;
- $H \in [L_p(\mathbb{R};X)]$.

In future, we strongly will use the following proposition.

Proposition 2.7 *Let X be a B-space $p \in (1,\infty)$. If $H \in [L_p(\mathbb{R};X)]$, then $m \in [L_p^0(I_0;X)]m \in [L_p(I_0;X)]$.*

Further details regarding these and related results can be found, for example, in the monograph [21]. In what follows, for function $f \in L_1(I_0;X)$, we denote by $\{\hat{f}_k\}_{k\in\mathbb{Z}}$ (also written as $\{t_k(f)\}_{k\in\mathbb{Z}}$) the sequence of its X-valued Fourier coefficients, given by

$$\hat{f}_k := t_k(f) := \frac{1}{2\pi}\int_J f(t)e^{-ikt}dt, k \in \mathbb{Z}.$$

In work [6], the following theorem is proved.

Theorem 2.8 *[6] Let $X \in UMDp \in (1,\infty)$. Then the exponential system $\mathcal{E} = \{e^{int}\}_{n\in\mathbb{Z}}$ forms t-basis for $L_p(I_0;X)$, i.e. $\forall f \in L_p(I_0;X)$ has a unique expansion in the form*

$$f(t) = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{int}, \tag{2.1}$$

in $L_p(I_0;X)$. Moreover, for $\forall m \in \mathbb{Z}$, the following series

$$(R_m^+ f)(t) = f_+(t) = \sum_{n=m}^{\infty} \hat{f}_n e^{int},$$
$$(R_m^- f)(t) = f_-(t) = \sum_{n=-\infty}^{m-1} \hat{f}_n e^{int},$$

also converges in $L_p(I_0;X)$ (so-called t-Riesz property) and $R_m^{\pm} \in [L_p(I_0;X)]$.

2.4 t-basicity of trigonometric system. t-Riesz property

Consider the following classical trigonometric system

$$\left\{\frac{1}{2}, \cos nx, \sin nx\right\}_{n\in\mathbb{N}}. \tag{2.2}$$

Let $f \in L_p(\mathcal{J}; X)$. Denote by $\{\hat{f}_n^{\,c}, \hat{f}_n^{\,s}\}_{n\in\mathbb{Z}_+}$ the Fourier coefficients of f on (2.2):

$$\left.\begin{aligned} v_n^c(f) = \hat{f}_n^{\,c} = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\, dt, \quad n \in \mathbb{Z}_+; \\ v_n^s(f) = \hat{f}_n^{\,s} = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\, dt, \quad n \in \mathbb{N}. \end{aligned}\right\} \tag{2.3}$$

and denote by

$$S_n(f)(t) = \frac{1}{2}\hat{f}_0^{\,c} + \sum_{k=1}^{n}\left(\hat{f}_k^{\,c}\cos kt + \hat{f}_k^{\,s}\sin kt\right), \quad k \in \mathbb{N},$$

the Fourier sums of f. Paying attention to the expression

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\, dt, \quad k \in \mathbb{Z},$$

we obtain

$$\begin{aligned} \tilde{S}_n(f)(t) = \sum_{k=-n}^{n}\hat{f}_k e^{ikt} &= \hat{f}_0 + \sum_{k=1}^{n}\hat{f}_k(\cos kt + i\sin kt) + \sum_{k=1}^{n}\hat{f}_{-k}(\cos kt - i\sin kt) \\ &= \hat{f}_0 + \sum_{k=1}^{n}\left[(\hat{f}_k + \hat{f}_{-k})\cos kt + i(\hat{f}_k - \hat{f}_{-k})\sin kt\right], \end{aligned}$$

and in result it is true the relations between Fourier coefficients

$$\hat{f}_0^{\,c} = 2\hat{f}_0, \quad \hat{f}_k^{\,c} = \hat{f}_k + \hat{f}_{-k}, \quad \hat{f}_k^{\,s} = i(\hat{f}_k - \hat{f}_{-k}), \quad k \in \mathbb{N}.$$

Conversely, it holds

$$\hat{f}_0 = \frac{1}{2}\hat{f}_0^{\,c}, \quad \hat{f}_k = \frac{1}{2}\left(\hat{f}_k^{\,c} - i\hat{f}_k^{\,s}\right), \quad \hat{f}_{-k} = \frac{1}{2}\left(\hat{f}_k^{\,c} + i\hat{f}_k^{\,s}\right), \quad k \in \mathbb{N}.$$

Consequently, from these relations and from Theorem 2.8 follows the validity of the following
Proposition 2.9 *Let X be UMD space. Then the trigonometric system (2.2) forms t-basis for $L_p(\mathcal{J}; X)$, $1 < p < +\infty$.*
It is obvious that $f(\cdot) \in L_p(I; X) \Leftrightarrow f(\cdot - \pi) \in L_p(\mathcal{J}; X)$, and it holds

$$\| f(\cdot) \|_{L_p(I;X)} = \| f(\cdot - \pi) \|_{L_p(\mathcal{J};X)}.$$

Taking into account this relation we obtain that the Proposition 2.9 is equivalent to the following

Proposition 2.10 *Let X be UMD space. Then the exponential system $\mathcal{E}$ and at the same time the trigonometric system (2.2) forms t-basis for $L_p(I;X)$, $1 < p < +\infty$.*

Let the system (2.2) forms t-basis for $L_p(I;X)$. Assume

$$L_p^c(I;X) = \overline{L_t[\{\cos nt\}_{n\in\mathbb{Z}_+}]};$$

$$L_p^s(I;X) = \overline{L_t[\{\sin nt\}_{n\in\mathbb{N}}]}.$$

Take $\forall f \in L_p(I;X)$ and represent it in the form

$$f(t) = f^+(t) + f^-(t),$$

where

$$\left.\begin{aligned} f^+(t) &= \tfrac{f(t)+f(-t)}{2} \quad - \text{even}, \\ f^-(t) &= \tfrac{f(t)-f(-t)}{2} \quad - \text{odd}, \end{aligned}\right\} \tag{2.4}$$

parts of f. It is evident that

$$f(\cdot) \in L_p(I;X) \Leftrightarrow f(-\cdot) \in L_p(I;X).$$

Paying attention to this relation from (2.4) follows $f^\pm \in L_p(I;X)$. Expanding $f(\cdot)$ on the basis (2.2) we have

$$f(t) = S(f) = \frac{1}{2}f_0{}^c + \sum_{k=1}^{\infty}(f_k{}^c \cos kt + f_k{}^s \sin kt).$$

Since $f^-(\cdot)$ is odd and $f^+(\cdot)$ is even functions we have

$$S_n^c(f) = \frac{1}{2}f_0{}^c + \sum_{k=1}^{n} f_k{}^c \cos kt = \frac{1}{2}v_0^c(f) + \sum_{k=1}^{n} v_k^c(f)\cos kt$$

$$= \frac{1}{2}v_0^c(f^+) + \sum_{k=1}^{n} v_k^c(f^+)\cos kt$$

$$+\frac{1}{2}v_0^c(f^-) + \sum_{k=1}^{n} v_k^c(f^-)\cos kt = v_0^c(f^+) + \sum_{k=1}^{n} v_k^c(f^+)\cos kt$$

$$= v_0^c(f^+) + \sum_{k=1}^{n}[v_k^c(f^+)\cos kt + v_k^s(f^+)\sin kt] = S_n(f^+), \quad \forall n \in \mathbb{Z}_+,$$

here we took into account that $v_k^s(f^+) = 0$, $\forall k \in \mathbb{N}$. From here we obtain the sequence $\{S_n^c(f)\}_{n\in\mathbb{Z}_+}$ converges in $L_p(I;X)$ (namely, $S_n^c(f) \to f^+, n \to \infty$).

Analogously, it follows

$$S_n^s(f)=\sum_{k=1}^{n} f_k^{\,s}\mathrm{sin}kt=\sum_{k=1}^{n} v_k^s(f)\mathrm{sin}kt=v_0^c(f^-)+\sum_{k=1}^{n} v_k^c(f^-)\mathrm{cos}kt+\sum_{k=1}^{n} v_k^s(f^-)\mathrm{sin}kt$$
$$=S_n(f^-),\quad \forall n\in\mathbb{N}$$

(where we took into account that $v_k^c(f^-)=0$, $\forall k\in\mathbb{Z}_+$). Consequently, $S_n^s(f)\to f^-$, $n\to\infty$, in $L_p(I;X)$, and let

$$S^c(f)=\frac{1}{2}v_0^c(f)+\sum_{k=1}^{\infty} v_k^c(f)\mathrm{cos}kt,$$

$$S^s(f)=\sum_{k=1}^{\infty} v_k^s(f)\mathrm{sin}kt.$$

So, it is true

$$f=S(f)=S^c(f)+S^s(f),\quad \forall f\in L_p(I;X). \tag{2.5}$$

It is not hard to see that $S^c(f)\in L_p^c(I;X)$ and $S^s(f)\in L_p^s(I;X)$. Moreover, it is true

$$L_p^c(I;X)\cap L_p^s(I;X)=\{0\}.$$

Then from (2.5) it follows that the subspaces $L_p^c(I;X)$ and $L_p^s(I;X)$ are complemented in $L_p(I;X)$ and it holds the direct sum

$$L_p(I;X)=L_p^c(I;X)\dotplus L_p^s(I;X). \tag{2.6}$$

Let's accept the following

Definition 2.11 *We will call that the t-basis (2.2) for $L_p(I;X)$ has t-Riesz property in $L_p(I;X)$, if there exists a constant $C>0$, such that*

$$\| S_n^c(f)\|_{L_p(I;X)}\le C\| f\|_{L_p(I;X)},$$

$$\| S_n^s(f)\|_{L_p(I;X)}\le C\| f\|_{L_p(I;X)},$$

$\forall f\in L_p(I;X)$, $\forall n\in\mathbb{N}$.

From above reasoning the following proposition follows.

Proposition 2.12 *Let X be UMD space. Then the t-basis (2.2) for $L_p(I;X)$, $1<p<+\infty$, has t-Riesz property in $L_p(I;X)$.*

3. Main results. Consider the following system

$$\{u_n^+(x);\ u_{n+1}^-(x)\}_{n\in\mathbb{Z}_+}, \tag{3.1}$$

where

$$u_n^+(x)=\mathrm{cos}nx,\quad u_{n+1}^-(x)=x\mathrm{sin}(n+1)x,\quad n\in\mathbb{Z}.$$

This system is a set of all root functions of the following nonlocal spectral problem

$$\left.\begin{array}{l} u''(x)+\lambda^2 u(x)=0, \quad 0<x<2\pi, \\ u(0)=u(2\pi), \quad u'(0)=0. \end{array}\right\}$$

The main characteristic of this problem is that it has infinitely many associated functions. We will investigate t-basis properties of the system (3.1) in $L_p(\mathcal{J};X)$.

3.1 *t*-biorthogonality property of the system (3.1)

Let

$$\varphi_0^+(x)=\frac{2\pi-x}{2\pi^2}; \quad \varphi_k^+(x)=\frac{2\pi-x}{\pi^2}\cos kx;$$

$$\varphi_k^-(x)=\frac{1}{\pi}\sin kx, \quad k\in\mathbb{N}.$$

Consider the following operators

$$\left.\begin{array}{l} t_k^+(f)=\int_0^{2\pi} f(x)\varphi_k^+(x)\,dx, \quad k\in\mathbb{Z}_+; \\ t_k^-(f)=\int_0^{2\pi} f(x)\varphi_k^-(x)\,dx, \quad k\in\mathbb{N}. \end{array}\right\} \tag{3.2}$$

It is obvious that there exist $C>0$, such that

$$\| t_k^{\pm}(f) \|_X \le C \| f \|_{L_p(\mathcal{J};X)}, \quad \forall f\in L_p(\mathcal{J};X), \qquad \forall k.$$

From here immediately follows $\{t_k^{\pm}\}\subset[L_p(\mathcal{J};X);X]$. For elementary tensors of the form $f_n^{\pm}(\cdot)=u_n^{\pm}(\cdot)\otimes a$, $a\in X$, we have

$$t_k^+(f_n^+)=\delta_{kn}a; \quad t_k^+(f_{n-1}^-)=0, \quad \forall k,n\in\mathbb{Z}_+;$$

$$t_k^-(f_{n+1}^+)=0, \quad t_k^-(f_n^-)=\delta_{kn}a, \quad \forall k,n\in\mathbb{N}.$$

Consequently, the system of operators $\{t_k^+;\ t_{k+1}^-\}_{k\in\mathbb{N}}$ is a t-biorthogonal to the system $\{u_k^+;\ u_{k+1}^-\}_{k\in\mathbb{N}}$, i.e. it is valid

Lemma 3.1 *The system of operators* $\{t_k^+,;\ t_{k+1}^-\}_{k\in\mathbb{N}}$ *and* $\{u_k^+;\ u_{k-1}^-\}_{k\in\mathbb{N}}$ *are t-biorthogonal systems with respect to* $L_p(\mathcal{J};X)$.

3.2 *t*-Completeness. From Proposition 2.5 follows that the algebraic product $C_0^\infty(\mathcal{J})\otimes X$ is dense in $L_p(\mathcal{J};X)$. Thus, it is sufficient to prove that every elementary tensor $g(\cdot)\otimes a$, where $g\in C_0^\infty(\mathcal{J})$ & $a\in X$, can be approximated by t-linear combination of the system (3.1) in $L_p(\mathcal{J};X)$. In turn, for this it is sufficient to prove that $g\in\overline{L[(3.1)]}$ (the closure is taken in $L_p(\mathcal{J})$). And this immediately follows from the results of the works [22, 23]. So, it is valid the following

Lemma 3.2 *The system* (3.1) *is t-dense in* $L_p(\mathcal{J};X)$, $1\le p<+\infty$.

3.3 t-basicity of the system (3.1). And now, let's prove the following main theorem.

Theorem 3.3 *Let X be UMD space. Then the system* (3.1) *forms t-basis for* $L_p(\mathcal{J};X)$, $1<p<+\infty$.

Proof. Paying attention to the Lemma 3.1 and Lemma 3.2, according to the t-basis criterion it is sufficient to prove that the projectors

$$P_{nm}(f)=\sum_{k=0}^{n} t_k^{+}(f)\cos kx+\sum_{k=1}^{m} t_k^{-}(f)\sin kx,$$

$\forall n\in\mathbb{Z}_+, \forall m\in\mathbb{N}$, are uniformly bounded, where the operators $\{t_k^{\pm}\}$ are defined by expressions (3.2). From these expressions we have

$$l_0^{+}(f)=\frac{1}{\pi^2}\vartheta_0^{c}(g),\quad l_k^{+}(f)=\frac{1}{\pi}\vartheta_k^{c}(g);$$

$$l_k^{-}(f)=\vartheta_k^{s}(f),\quad \forall k\in\mathbb{N},$$

where $g(x)=(2\pi-x)f(x)$, $x\in\mathcal{J}$, and $\{\vartheta_k^{c}(\cdot);\vartheta_k^{s}(\cdot)\}$ are Fourier coefficients on the system (2.2). Taking into account t-Riesz property of the system (2.2) in $L_p(\mathcal{J};X)$, we obtain

$$\| P_{nm}(f)\|_{L_p(\mathcal{J};X)}\le\frac{1}{\pi^2}\|\vartheta_0^{c}(g)\|_{L_p(\mathcal{J};X)}$$
$$+\frac{1}{\pi}\left\|\sum_{k=1}^{n}\vartheta_k^{c}(g)\cos kx\right\|_{L_p(\mathcal{J};X)}+\left\|\sum_{k=1}^{n}\vartheta_k^{s}(f)\sin kx\right\|_{L_p(\mathcal{J};X)}$$
$$\le \text{const}\,(\| g\|_{L_p(\mathcal{J};X)}+\| f\|_{L_p(\mathcal{J};X)})\le \text{const}\,\| f\|_{L_p(\mathcal{J};X)},\quad \forall n\in\mathbb{Z}_+,\qquad \forall m\in\mathbb{N}.$$

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Data availability
No datasets were generated or analyzed during the current study.

Declarations

Conflict of interest
Bilal Bilalov is a member of the Editorial Board of this journal. He was not involved in the editorial handling, peer review, or decision-making process of this manuscript. The editorial process was handled independently. The authors declare no conflict of interest.

Author contributions
B.B. (Bilalov Bilal) and S.S. (Sadigova Sabina) jointly conceptualized the study, developed the methodology, conducted the investigation, analyzed the results, and contributed to writing and revising the manuscript. B.B. prepared the figures and performed the primary analysis, while S.S. supervised the research and validated the findings. Both authors read and approved the final manuscript.

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