



<https://doi.org/10.58225/sw.2025.2-73-78>

ON MODIFIED ORLICZ-MORREY SPACES

Ali Akbulut¹, Vagif Guliyev^{1,2,3}, Mehriban Omarova^{3,4}

¹Kirsehir Ahi Evran University, Kirsehir, Turkey, orcid ID 0000-0002-1435-071X

²Azerbaijan University of Architecture and Construction, Baku, Azerbaijan,
orcid ID 0000-0001-7486-0298

³Institute of Mathematics and Mechanics, Baku, Azerbaijan

⁴Baku State University, Baku, Azerbaijan, mehriban_omarova@yahoo.com,
orcid ID 0009-0008-2301-3273

Abstract. In this article, we discussed the third version of weak modified Orlicz-Morrey spaces, which is an enlargement of the third version of (strong) modified Orlicz-Morrey spaces. Similar to its first version and second version, the third version of strong and weak modified Orlicz-Morrey space is considered as a generalization of strong and weak Orlicz spaces and strong and weak modified Morrey spaces. This study investigated some properties of the third version modified Orlicz-Morrey spaces. AMS Mathematics Subject Classification: 42B20, 42B25, 42B35; 46E30

Keywords: Orlicz spaces; modified Morrey spaces; modified Orlicz-Morrey spaces; third version

1.Introduction. As is well known, Morrey spaces are widely used to investigate the local behavior of solutions to second order elliptic partial differential equations (PDE). Recall that the classical Morrey spaces $M^{p,\lambda}(R^n)$ and the modified Morrey spaces $\widehat{M}^{p,\lambda}(R^n)$ are defined as follows:

$$M^{p,\lambda}(R^n) = \left\{ f \in L^p_{loc}(R^n) : \|f\|_{M^{p,\lambda}} = \sup_{x \in R^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))} < \infty \right\}$$

and

$$\widehat{M}^{p,\lambda}(R^n) = \left\{ f \in L^p_{loc}(R^n) : \|f\|_{\widehat{M}^{p,\lambda}} = \sup_{x \in R^n, r > 0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))} < \infty \right\}$$

respectively, where $[r]_1 = \min\{1, r\}$, $0 \leq \lambda \leq n$, $1 \leq p < \infty$. The space $M^{p,\lambda}(R^n)$ is an extension of the space $L^p(R^n)$ in the sense that $M^{p,0}(R^n) = L^p(R^n)$ and $M^{p,n}(R^n) = L^\infty(R^n)$. Moreover, $\widehat{M}^{p,\lambda}(R^n) = M^{p,\lambda}(R^n) \cap L^p(R^n)$, see [1, 4, 5, 6]. Here and everywhere in the sequel, $B(x, r)$ is the ball in R^n of radius r centered at x and $|B(x, r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in R^n .

By $WM^{p,\lambda}(R^n)$ and $\widehat{WM}^{p,\lambda}(R^n)$ we denote the weak Morrey space and the weak modified Morrey space defined as the set of functions f in the local weak space $WL^p_{loc}(R^n)$ for which

$$\|f\|_{WM^{p,\lambda}} = \sup_{x \in R^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,r))} < \infty$$

and

$$\|f\|_{WM^{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r > 0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,r))} < \infty$$

respectively, where $\|f\|_{WL^p(B(x,r))} = \sup_{t>0} t \left| \left\{ y \in B(x,r) : |f(y)| > t \right\} \right|^{\frac{1}{p}}$.

Orlicz spaces are a generalization of Lebesgue spaces L^1 , see [12, 13]. They are useful tools to study harmonic analysis and its applications. For example, the Hardy-Littlewood maximal operator is bounded on L^p for $p > 1$, see [9, 10] and [2] for more precise statements. On the other hand, Morrey spaces was introduced by Ch. B. Morrey to estimate solutions of partial differential equations. After that there are many papers about Morrey spaces. For example, the boundedness of the Hardy-Littlewood maximal operator, fractional integral operators and smoothness properties of solutions of PDE are studied by many mathematicians.

In [3], the generalized Orlicz-Morrey space $M^{\Phi,\varphi}(\mathbb{R}^n)$ was introduced to unify Orlicz spaces and generalized Morrey spaces. Other definitions of generalized Orlicz-Morrey spaces can be found in [11, 14]. In works of [7], generalized Orlicz-Morrey space is the third kind and the ones in [11] and [14] are the first kind and the second kind, respectively. Notice that the definition of the space of the third kind relies only on the fact that $L^{\Phi}(\mathbb{R}^n)$ is a normed linear space, which is independent of the condition that it is generated by modulars. These spaces are one of the important topics in mathematical analysis, particularly in harmonic analysis. There are three versions of Orlicz-Morrey spaces, i.e: Nakai's, Sawano-Sugano-Tanaka's (Gala et al., 2015) and Deringoz-Guliyev-Samko's (Deringoz et al., 2015) versions.

There are three versions of Orlicz-Morrey spaces. In this article, we focus on the third version of the modified Orlicz-Morrey and the weaker modified Orlicz-Morrey spaces, which extend the third version of the classical Orlicz-Morrey spaces. As in the first and second versions, the third version of the strong and weak Orlicz-Morrey spaces serves as a unifying framework that generalizes the strong and weak Orlicz spaces, the strong and weak Morrey spaces, and their generalized counterparts.

In this manuscript, the authors investigate the third version of weak modified Orlicz-Morrey spaces, which extends the corresponding third version of modified Orlicz-Morrey spaces. Similar to the first and second versions, the third versions of both strong and weak modified Orlicz-Morrey spaces serve as generalizations of the classical strong and weak Orlicz spaces as well as the strong and weak modified Morrey spaces. This study examines several fundamental properties of the third version of modified Orlicz-Morrey spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 On Young functions and Orlicz spaces. We recall the definition of Young functions.

Definition 2.1. A function $\Phi : [0, \infty) \rightarrow [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r \rightarrow +0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r \rightarrow \infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. The set of Young functions such that $0 < \Phi(r) < \infty$ for $0 < r < \infty$

will be denoted by Y . If $\Phi \in Y$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself. For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf \{ r \geq 0 : \Phi(r) > s \}.$$

If $\Phi \in Y$, then Φ^{-1} is the usual inverse function of Φ . We note that

$$\Phi(\Phi^{-1}(r)) \leq r \leq \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \leq r < \infty.$$

It is well known that

$$r \leq \Phi^{-1}(r) \tilde{\Phi}^{-1}(r) \leq 2r \quad \text{for } r \geq 0, \quad (2.1)$$

where $\tilde{\Phi}(r)$ is defined by

$$\tilde{\Phi}(r) = \begin{cases} \sup \{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2 -condition, denoted also as

$$\Phi \in \Delta_2 \quad \text{if} \\ \Phi(2r) \leq C\Phi(r), \quad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in Y$. A Young function Φ is said to satisfy the ∇_2 -condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2C} \Phi(kr), \quad r \geq 0,$$

for some $C > 1$.

A Young function Φ is said to satisfy the Δ' -condition, denoted also as

$$\Phi \in \Delta', \quad \text{if} \\ \Phi(tr) \leq C\Phi(t)\Phi(r), \quad t, r > 0$$

for some $C > 1$.

Note that, each element of Δ' -class is also an element of Δ_2 -class.

Remark 2.2. Let $\Phi \in \Delta'$, then we have

$$\Phi(tr) \leq C\Phi(t)\Phi(r), \quad t, r \geq 0.$$

If we set $\Phi(t) = u$ and $\Phi(r) = v$ we get $\Phi(\Phi^{-1}(u)\Phi^{-1}(v)) \leq Cuv \Rightarrow \Phi^{-1}(u)\Phi^{-1}(v) \leq \Phi^{-1}(Cuv) \leq C\Phi^{-1}(uv)$, since $\Phi \in Y$ and Φ^{-1} is concave.

Definition 2.3. (Orlicz Space). For a Young function Φ , the set

$$L^\Phi(R^n) = \left\{ f \in L^1_{loc}(R^n) : \int_{R^n} \Phi(k|f(x)|) dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L^\Phi(R^n) = L^p(R^n)$. If $\Phi(r) = 0$, $(0 \leq r \leq 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^\Phi(R^n) = L^\infty(R^n)$. The space $L^\Phi_{loc}(R^n)$ is defined as the set of all functions f such that $f\chi_B \in L^\Phi(R^n)$ for all balls $B \subset R^n$. $L^\Phi(R^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_{R^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

We note that

$$\int_{R^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L^\Phi}}\right) dx \leq 1 \quad (2.2)$$

Lemma 2.4. [3] For a Young function Φ and $B = B(x, r)$, the following inequality is valid

$$\|f\|_{L^1(B)} \leq 2|B|\Phi^{-1}(|B|^{-1})\|f\|_{L^\Phi(B)}.$$

Definition 2.5. Let Φ be a Young function. The weak Orlicz space is defined as

$$WL^\Phi(R^n) := \left\{ f \in L^1_{loc}(R^n) : \|f\|_{WL^\Phi} < \infty \right\},$$

where

$$\|f\|_{WL^\Phi} = \inf \left\{ \lambda > 0 : \sup_{t>0} \Phi\left(\frac{t}{\lambda}\right) d_f(t) \leq 1 \right\},$$

and $d_f(t) = \left| \left\{ x \in R^n : |f(x)| > t \right\} \right|$.

Lemma 2.6. If $0 < \|f\|_{WL^\Phi} < \infty$, then

$$\sup_{t>0} \Phi\left(\frac{t}{\|f\|_{WL^\Phi}}\right) d_f(t) \leq 1. \quad (2.3)$$

Proof. By the definition of $\|\cdot\|_{WL^\Phi}$, for all $\lambda > \|f\|_{WL^\Phi}$ we have $\Phi\left(\frac{t}{\lambda}\right) d_f(t) \leq 1, \forall t > 0$. Now as λ decreases to $\|f\|_{WL^\Phi}$, the quotient $\frac{t}{\lambda}$ increases to $\frac{t}{\|f\|_{WL^\Phi}}$. By the left-continuity of Φ , we have

$\Phi\left(\frac{t}{\lambda}\right) \uparrow \Phi\left(\frac{t}{\|f\|_{WL^\Phi}}\right)$. Therefore we get the desired result. By elementary calculations we have the following.

Lemma 2.7. Let Φ be a Young function and B a set in R^n with finite Lebesgue measure. Then

$$\|\chi_B\|_{L^\Phi} = \|\chi_B\|_{WL^\Phi} = \frac{1}{\Phi^{-1}(|B|^{-1})}.$$

The following theorem is an analogue of the Lebesgue differentiation theorem in Orlicz spaces.

Theorem 2.8. [8] Suppose that Φ is a Young function and let $f \in L^\Phi(R^n)$ be nonnegative. Then

$$\liminf_{r \rightarrow 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} \geq f(x), \quad \text{for almost every } x \in R^n.$$

If we moreover assume that $\Phi \in \Delta'$, then

$$\liminf_{r \rightarrow 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} = f(x), \quad \text{for almost every } x \in R^n.$$

3 Modified Orlicz-Morrey Spaces

Definition 3.1. For a Young function Φ and $\lambda \in R$, we denote by $M^{\Phi,\lambda}(R^n)$ the Orlicz-Morrey space, defined as the space of all functions $f \in L_{loc}^\Phi(R^n)$ with finite quasinorm

$$\|f\|_{M^{\Phi,\lambda}} = \sup_{x \in R^n, r>0} \Phi^{-1}(r^{-\lambda}) \|f\chi_{B(x,r)}\|_{L^\Phi}.$$

Note that $M^{\Phi,\lambda}(R^n)_{\lambda=0} = L^\Phi(R^n)$ and $M^{\Phi,\lambda}(R^n)_{\Phi(t)=t^p} = M^{p,\lambda}(R^n)$.

Definition 3.2. For a Young function Φ and $\lambda \in R$, we denote by $\widehat{M}^{\Phi,\lambda}(R^n)$ the modified Orlicz-Morrey space, defined as the space of all functions $f \in L_{loc}^\Phi(R^n)$ with finite quasinorm

$$\|f\|_{\widehat{M}^{\Phi,\lambda}} = \sup_{x \in R^n, r>0} \Phi^{-1}([r]_1^{-\lambda}) \|f\chi_{B(x,r)}\|_{L^\Phi}.$$

Note that $\widehat{M}^{\Phi,\lambda}(R^n)_{\lambda=0} = L^\Phi(R^n)$ and $\widehat{M}^{\Phi,\lambda}(R^n)_{\Phi(t)=t^p} = \widehat{M}^{p,\lambda}(R^n)$.

Definition 3.3. For a Young function Φ and $\lambda \in R$, we denote by $WM^{\Phi,\lambda}(R^n)$ the weak Orlicz-Morrey space, defined as the space of all functions $f \in WL_{loc}^\Phi(R^n)$ with finite quasinorm

$$\|f\|_{WM^{\Phi,\lambda}} = \sup_{x \in R^n, r>0} \Phi^{-1}\left(|B(x,r)|^{-\frac{\lambda}{n}}\right) \|f\chi_{B(x,r)}\|_{WL^\Phi}.$$

Definition 3.4. For a Young function Φ and $\lambda \in R$, we denote by $LM^{\Phi,\lambda}(R^n)$ the local

Orlicz-Morrey space, defined as the space of all functions $f \in L_{loc}^\Phi(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM^{\Phi,\lambda}} = \sup_{r>0} \Phi^{-1}(r^{-n}) \|f\chi_{B(0,r)}\|_{L^\Phi}.$$

Note that $LM^{\Phi,\lambda}(\mathbb{R}^n)|_{\lambda=0} = L^\Phi(\mathbb{R}^n)$ and $LM^{\Phi,\lambda}(\mathbb{R}^n)|_{\Phi(t)=t^p} = LM^{p,\lambda}(\mathbb{R}^n)$.

The following statement is true.

Lemma 3.5. If $\Phi \in \Delta'$, then $\widehat{M}^{\Phi,n}(\mathbb{R}^n) = L^\Phi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and

$$\|f\|_{\widehat{M}^{\Phi,n}} \approx \max\{\|f\|_{L^\infty}, \|f\|_{L^\Phi}\}$$

Proof. Let $f \in L^\Phi(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned} \|f\|_{\widehat{M}^{\Phi,n}} &= \sup_{x \in \mathbb{R}^n, r>0} \Phi^{-1}([r]^{-n}) \|f\chi_{B(x,r)}\|_{L^\Phi} = \max\left\{ \sup_{x \in \mathbb{R}^n, 0<r \leq 1} \Phi^{-1}([r]^{-n}) \|f\chi_{B(x,r)}\|_{L^\Phi}, \sup_{x \in \mathbb{R}^n, r>1} \Phi^{-1}([r]^{-n}) \|f\chi_{B(x,r)}\|_{L^\Phi} \right\} \\ &= \max\left\{ \sup_{x \in \mathbb{R}^n, 0<r \leq 1} \Phi^{-1}(r^{-n}) \|f\chi_{B(x,r)}\|_{L^\Phi}, \Phi^{-1}(1) \sup_{x \in \mathbb{R}^n, r>1} \|f\chi_{B(x,r)}\|_{L^\Phi} \right\} \\ &\leq \max\left\{ \|f\|_{L^\infty} \sup_{x \in \mathbb{R}^n, 0<r \leq 1} \Phi^{-1}(r^{-n}) \|\chi_{B(x,r)}\|_{L^\Phi}, \Phi^{-1}(1) \|f\|_{L^\Phi} \right\} \\ &\leq \max\{\|f\|_{L^\infty}, \|f\|_{L^\Phi}\}, \end{aligned}$$

which implies $\|f\|_{\widehat{M}^{\Phi,n}} \leq \max\{\|f\|_{L^\infty}, \|f\|_{L^\Phi}\}$.

Now let $f \in \widehat{M}^{\Phi,n}(\mathbb{R}^n)$. Theorem 2.8 implies that $|f(x)| \leq \|f\|_{\widehat{M}^{\Phi,n}}$ for almost every $x \in \mathbb{R}^n$, which means that $\|f\|_{\widehat{M}^{\Phi,n}} \geq \|f\|_{L^\infty}$. It also follows from the definition of modified Orlicz-Morrey spaces that $\|f\|_{L^\Phi} \leq \|f\|_{\widehat{M}^{\Phi,n}}$.

Therefore,

$$\max\{\|f\|_{L^\infty}, \|f\|_{L^\Phi}\} \leq \|f\|_{\widehat{M}^{\Phi,n}}.$$

In the following we denote by Θ the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 3.6. Let Φ be a Young function. If $\lambda < 0$ or $\lambda > n$ and $\Phi \in \Delta'$, then $M^{\Phi,\lambda}(\mathbb{R}^n) = \Theta$.

Proof. First let $\lambda < 0$ and $f \in M^{\Phi,\lambda}(\mathbb{R}^n)$. For all $x \in \mathbb{R}^n$ and $r > 0$ we have

$$\|f\chi_{B(0,r)}\|_{L^\Phi} \leq \frac{\|f\|_{M^{\Phi,\lambda}}}{\Phi^{-1}(r^{-\lambda})},$$

which implies that $\|f\|_{L^\Phi} = \lim_{r \rightarrow \infty} \|f\chi_{B(0,r)}\|_{L^\Phi} = 0 \Rightarrow f(x) = 0$, for almost every $x \in \mathbb{R}^n$, since $\lim_{r \rightarrow \infty} \Phi^{-1}(r^{-\lambda}) = \infty$.

Now let $\lambda > n$ and $f \in M^{\Phi,\lambda}(\mathbb{R}^n)$. For all $x \in \mathbb{R}^n$ and $r > 0$ we have from Remark 2.2 and Theorem 2.8

$$\frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} \leq \frac{\Phi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n}r^{n-\lambda})} \|f\|_{M^{\Phi,\lambda}} \leq \frac{\|f\|_{M^{\Phi,\lambda}}}{\Phi^{-1}(r^{n-\lambda})},$$

which implies that $|f(x)| = 0$, for almost every $x \in \mathbb{R}^n$.

Remark 3.7. In the case $\Phi(t) = t^p$, for $1 \leq p < \infty$ from Lemmas 3.5 and 3.6 we get the following well-known results: $M^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ and $\widehat{M}^{p,\lambda}(\mathbb{R}^n) = \Theta$ for $\lambda < 0$ or $\lambda > n$.

References

1. Aykol C, Armutcu H, Omarova MN. Maximal commutator and commutator of maximal function on modified Morrey spaces. Transactions of the National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences. Mathematics. 2016;36(1):29–35
2. Cianchi A. Strong and weak type inequalities for some classical operators in Orlicz spaces. Journal of the London Mathematical Society. 1999;60(1):187–202
3. Deringoz F, Guliyev VS, Samko S. Boundedness of maximal and singular operators on generalized Orlicz–Morrey spaces. In: Operator Theory, Operator Algebras and Applications. Operator Theory: Advances and Applications. Vol. 242. Basel: Birkhäuser; 2014. p. 139–158
4. Guliyev VS. Integral operators on function spaces on homogeneous groups and on domains in $\mathbb{R}^n$. Doctoral dissertation. Moscow: Steklov Mathematical Institute; 1994. 329 p. (In Russian)
5. Guliyev VS. Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications. Baku: Casioglu; 1999. 332 p. (In Russian)
6. Guliyev VS, Hasanov JJ, Zeren Y. Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces. Journal of Mathematical Inequalities. 2011;5(4):491–506
7. Guliyev VS, Hasanov SG, Sawano Y, Noi T. Non-smooth atomic decompositions for generalized Orlicz–Morrey spaces of the third kind. Acta Applicandae Mathematicae. 2016;145:133–174
8. Hencl S, Kleprlik L. Composition of qqq-quasiconformal mappings and functions in Orlicz–Sobolev spaces. Illinois Journal of Mathematics. 2012;56(3):931–955
9. Kita H. On maximal functions in Orlicz spaces. Proceedings of the American Mathematical Society. 1996;124:3019–3025
10. Kita H. On Hardy–Littlewood maximal functions in Orlicz spaces. Mathematische Nachrichten. 1997;183:135–155
11. Nakai E. Generalized fractional integrals on Orlicz–Morrey spaces. In: Banach and Function Spaces (Kitakyushu, 2003). Yokohama: Yokohama Publishers; 2004. p. 323–333
12. Orlicz W. Über eine gewisse Klasse von Räumen vom Typus B. Bulletin de l'Académie Polonaise des Sciences. Série A. 1932:207–220; reprinted in: Collected Papers. Warsaw: PWN; 1988. p. 217–230
13. Orlicz W. Über Räume (LM). Bulletin de l'Académie Polonaise des Sciences. Série A. 1936:93–107; reprinted in: Collected Papers. Warsaw: PWN; 1988. p. 345–359.
14. Sawano Y, Sugano S, Tanaka H. Orlicz–Morrey spaces and fractional operators. Potential Analysis. 2012;36(4):517–556

Received: 01/09/2025 Accepted: 16/12/2025 Published: 30/12/2025

© The Author(s) 2025. Published by AzUAC. LTD. This is an open access article under the CC BY License (<https://creativecommons.org/licenses/by/4.0/>).