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STABILITY OF FUNCTIONALLY GRADED CYLINDERS UNDER HYDROSTATIC PRESSURE WITH FIXED BOUNDARY CONDITIONS RESTING ON ELASTIC FOUNDATIONS

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Abstract. In this study, the buckling problem of cylindrical shells composed of functionally graded materials (FGMs) under hydrostatic pressure and resting on a Pasternak-type elastic foundation is investigated. First, the fundamental relations of functionally graded (FG) cylindrical shells and the Pasternak elastic foundation model are mathematically formulated, and then the governing equations are derived. By seeking appropriate shape functions satisfying fixed boundary conditions, the Galerkin procedure is applied to solve the resulting partial differential equations. An analytical expression for the dimensionless critical hydrostatic pressure (DCHP) of FG cylindrical shells with fixed edges on a Pasternak elastic foundation is obtained. This expression is minimized with respect to the buckling mode, and the minimum values of DCHP are calculated numerically both in the presence and absence of the elastic foundation. Considering different volume fraction indices and foundation parameters, the minimum values of DCHP are determined, followed by numerical analyses, interpretations, and generalizations.

Keywords: functionally graded material, fixed boundary conditions, cylindrical shells, stability, critical hydrostatic pressure, elastic foundations

1. Introduction. Composite materials have long attracted significant interest from researchers due to their superior properties such as low weight, high strength, and resistance to corrosion and wear. Owing to these advantages, they are widely used across a range of engineering applications. However, one of the main drawbacks of traditional composite materials is the occurrence of sudden changes in mechanical properties at the interfaces between different constituents, which can lead to crack initiation and propagation. To address this issue, a new class of composite materials known as Functionally Graded Materials (FGMs) was introduced by Japanese material scientists in 1984 as part of a space research project. FGMs enable a smooth and continuous variation of physical, chemical, and mechanical properties—such as elastic modulus, Poisson's ratio, and density—across the material volume [1].

In the literature, two principal modeling approaches are commonly employed to represent FGMs. In the first approach, the volume fraction of the ceramic or metallic phase is assumed to be a piecewise continuous function. Accordingly, the FGM structure is modeled as a series of sub-layers, each with constant volume fractions, effectively treating the material as composed of semi-homogeneous ceramic-metal layers. In this way, material properties remain uniform within each layer while transitioning gradually between layers, enabling a simplified yet practical modeling framework for engineering analyses.

In contrast, the second approach assumes a continuously varying volume fraction of the constituent phases across the thickness of the structure, resulting in a more realistic representation of material heterogeneity. In this method, the ceramic volume fraction is defined as a continuous function

of the through-thickness coordinate, allowing the modeling of FGMs with a smooth gradation profile instead of discrete layers [2, 3].

FGMs have demonstrated considerable potential in various engineering domains, and interest in their mechanical behavior and potential applications continues to grow [4]. The advantages of FGMs necessitate thorough investigations into the structural responses of FGM-based components under different elastic foundations, loading conditions, and boundary constraints, and integrating these responses into engineering design practices. In this context, a substantial body of research has been dedicated to the stability and vibration analysis of cylindrical shells composed of FGMs [5–9].

However, most existing studies have focused on simply supported boundary conditions without considering the influence of the elastic foundation, relying primarily on analytical or numerical methods. In many advanced industrial applications, FGM cylindrical shells are embedded within elastic media, where the foundation effect plays a non-negligible role in their structural behavior. These surrounding media can be effectively modeled using various parameters, among which the two-parameter Pasternak foundation model is frequently employed due to its ability to account for both shear interaction and normal stiffness [10].

In recent years, growing attention has been directed towards the stability behavior of fixed FGM cylindrical shells under external pressure while embedded in elastic media [11–13]. Such structures are widely used in advanced engineering applications including aerospace and aeronautical engineering, nuclear energy systems, biomedical implants, automotive industry, and thermal protection systems. Accurate modeling of their stability behavior under fixed boundary conditions and external pressure is crucial for ensuring structural safety, longevity, and performance. Furthermore, the inclusion of elastic foundation effects enables more realistic designs for systems involving soil-structure interaction, embedded structures, and elastomer-supported configurations.

A review of the current literature reveals that analytical studies addressing the buckling behavior of fixed FGM cylindrical shells under hydrostatic pressure in an elastic medium are still relatively limited. This study aims to address this research gap and contribute to the growing body of knowledge in the field. In addition, it provides a theoretical modeling framework that can serve as a foundation for the development of design criteria applicable to various engineering disciplines.

2. Formulation of the problem. In this study, a functionally graded cylindrical shell characterized by thickness t , radius r , and length a is examined under the assumption that it is supported by a Pasternak-type elastic foundation, as schematically illustrated in Fig. 1. A Cartesian coordinate system $Oxyz$ is introduced such that the origin OOO coincides with the left boundary of the shell's mid-surface. The Oxx -axis is directed along the shell's longitudinal axis, whereas the Oy -axis follows the circumferential direction. The Oz -axis is defined as the outward normal to the mid-surface generated by the Ox – Oy plane, oriented radially inward toward the geometric center of the cylindrical configuration. This coordinate convention permits a systematic representation of both the kinematic field and the associated deformation measures.

The elastic foundation considered in this work incorporates the Pasternak formulation, which accounts for both normal Winkler-type resistance and the shear interaction between adjacent foundation springs. Within this analytical framework, the governing expression of the Pasternak foundation response is introduced in the following form [9]:

$$R = k_1 w - k_2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \quad (1)$$

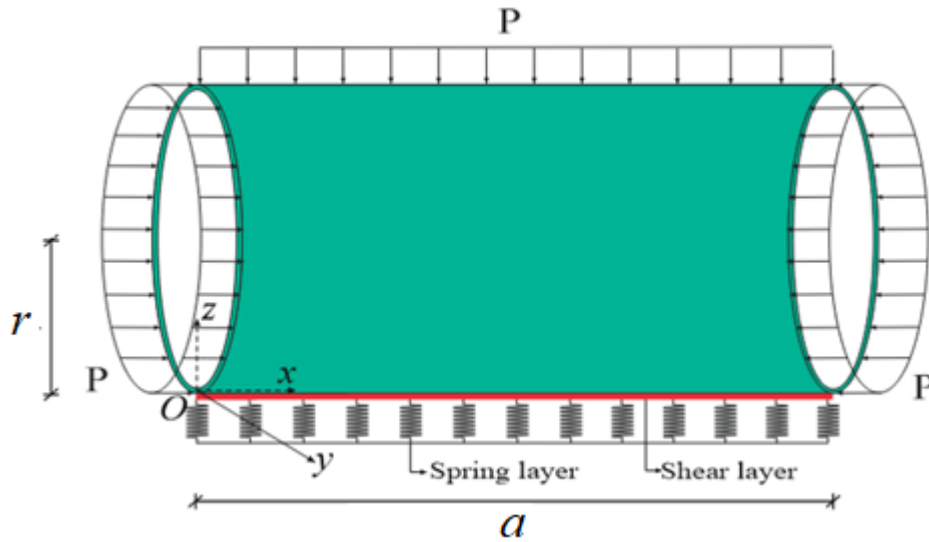


Fig. 1. FG cylindrical shell resting on a Pasternak elastic foundation under hydrostatic pressure and the associated coordinate system

In this formulation, R denotes the distributed reaction generated by the Pasternak–Winkler elastic substrate. The parameter k_1 (N/m³) corresponds to the classical Winkler modulus, representing the compressive stiffness of the foundation in the transverse direction, whereas k_2 (N/m) characterizes the shear stiffness associated with an idealized, exceedingly thin intermediate layer that permits shear interaction between adjacent foundation points. The quantity w signifies the transverse displacement of the shell mid-surface, assumed to remain small compared with the shell thickness and measured normal to the reference surface. Collectively, this two-parameter foundation model captures the combined influence of vertical resistance and shear coupling, thereby providing a more realistic representation of the substrate response than the Winkler model alone.

Functionally graded materials are commonly characterized through either power-law or exponential gradation schemes. In the present analysis, the cylindrical shell is assumed to follow a power-law variation in its material composition. Accordingly, the equivalent elastic parameters of the shell—specifically the effective Young’s modulus and Poisson’s ratio—are formulated as given below [11]:

$$E_z^{fg} = \left(\frac{2z+t}{2t} \right)^N (E_c - E_m) + E_m, \quad \nu_z^{fg} = \left(\frac{2z+t}{2t} \right)^N (\nu_c - \nu_m) + \nu_m \quad (2)$$

Assuming that the functionally graded cylindrical shell is subjected to a spatially uniform hydrostatic load, the corresponding membrane force relations—formulated without invoking the first moment of stresses about the reference surface—can be expressed in the following form:

$$N_{11}^0 = -0.5 \cdot r \cdot P, \quad N_{12}^0 = 0, \quad N_{22}^0 = -r \cdot P \quad (3)$$

3. Governing equations and solution procedure. The governing equations for cylindrical shells made of FGMs can be expressed within the framework of the Kirchhoff–Love shell theory can be obtained as follows [8]:

$$S_1 \frac{\partial^4 \Psi}{\partial x^4} + 2(S_2 - S_5) \frac{\partial^4 \Psi}{\partial x^2 \partial y^2} + S_1 \frac{\partial^4 \Psi}{\partial y^4} + \frac{1}{r} \frac{\partial^2 \Psi}{\partial x^2} - S_3 \frac{\partial^4 w}{\partial x^4} - 2(S_4 + S_6) \frac{\partial^4 w}{\partial x^2 \partial y^2} - S_3 \frac{\partial^4 w}{\partial y^4} + \frac{1}{r} \frac{\partial^2 w}{\partial x^2} - 0.5r \cdot P \frac{\partial^2 w}{\partial x^2} - r \cdot P \frac{\partial^2 w}{\partial y^2} - k_1 w + k_2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0 \quad (4)$$

$$T_1 \frac{\partial^4 \Psi}{\partial x^4} + 2(T_2 + T_5) \frac{\partial^4 \Psi}{\partial x^2 \partial y^2} + T_1 \frac{\partial^4 \Psi}{\partial y^4} - T_4 \frac{\partial^4 w}{\partial x^4} - 2(T_3 - T_6) \frac{\partial^4 w}{\partial x^2 \partial y^2} - T_{14} \frac{\partial^4 w}{\partial y^4} + \frac{1}{r} \frac{\partial^2 w}{\partial x^2} = 0 \quad (5)$$

In this context, the coefficients $S_k, T_k (k=1,2,...,6)$ are parameters determined by the material gradation characteristics of the FGM and the geometric and mechanical properties of the cylindrical shell.

At both longitudinal ends of the FG cylindrical shell, fixed boundary conditions are assumed to be valid [3, 8, 13]. The following approximating functions, corresponding to the deflection and the Airy stress function, are constructed to satisfy the fixed boundary conditions [8]:

$$w = C_1 \sin^2 \alpha_1 x \cos \alpha_2 y, \quad \Psi = C_2 \sin^2 \alpha_1 x \cos \alpha_2 y \quad (6)$$

Here, $\alpha_1 = \frac{m\pi}{a}$ and $\alpha_2 = \frac{n\pi}{r}$ are parameters, while m and n denote the longitudinal and circumferential wave numbers, respectively. The coefficients $C_j (j=1,2)$ represent the unknown amplitudes associated with the deflection and the Airy stress function.

For FG cylindrical shells resting on a Pasternak foundation and subjected to hydrostatic pressure, the strain compatibility and stability equations defined by (6) and (7) are multiplied by a weight function over the relevant region $\Lambda = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq 2\pi r\}$, and the Galerkin procedure is applied. As a result of this process, the following analytical expression is obtained for the critical hydrostatic pressure of the fixed FG cylindrical shells:

$$P_{wp}^{Hcr} = \frac{1}{(2\alpha_1^2 + 3\alpha_2^2)r} \left\{ 16C_3\alpha_1^4 + 8\alpha_1^2\alpha_2^2(C_4 + C_6) + 3C_3\alpha_2^2 - \left[16C_2\alpha_1^4 + 8\alpha_1^2\alpha_2^2(C_1 - C_5) + 3C_2\alpha_2^4 - \frac{4\alpha_1^2}{r} \right] \right. \\ \left. \times \frac{16B_4\alpha_1^4 + 4\alpha_1^2\alpha_2^2(2B_3 - B_6) + 3B_4\alpha_2^4 + \frac{4\alpha_1^2}{r}}{16B_1\alpha_1^4 + 8\alpha_1^2\alpha_2^2(2B_2 + B_5) + 3B_1\alpha_2^4} + 3k_1 + k_2(4\alpha_1^2 + 3\alpha_2^2) \right\} = 0 \quad (7)$$

The expression given below is employed for evaluating the DCHP of FG cylindrical shells with both ends fixed and supported by a Pasternak elastic foundation:

$$P_{1wp}^{Hcr} = \frac{P_{wp}^{Hcr}}{E_c} \quad (8)$$

Equation (7) yields a particular case representing the critical hydrostatic pressure for EG cylindrical shells without foundation support when the foundation parameters k_1 and k_2 are set to zero.

Minimization of equation (7) with respect to the mode pairs (m, n) yields the minimum values of the DCHP. For the numerical computation of these values, the Maple software is utilized.

4. Results and discussions. In numerical analyses, two different types of cylindrical shells with distinct FG profiles, denoted as FG₁ and FG₂, are considered. These functionally graded materials are composed of mixtures of metal and ceramic phases. Specifically, the cylindrical shell composed of Si₃N₄ (silicon nitride) and SUS304 stainless steel is referred to as FG₁, while the shell formed from Ti-6Al-4V titanium alloy and ZrO₂ (zirconia) is referred to as FG₂. The mechanical properties of these materials and their constituent components are adopted from the study by Shen [13]. The geometric parameters of the cylindrical shell are taken as follows: $r/t = 75$ ve $a/r = 2$.

Table 1 presents the variation of the DCHP values for the FG₁-quad and FG₂-quad profiles as a function of the elastic foundation parameters k_1 and k_2 . When comparing DCHP values of cylindrical shells with FG₁-quad and FG₂-quad profiles to those of homogeneous Si₃N₄ and ZrO₂ cylindrical shells, it is observed that in the absence of a foundation, the FG₁ and FG₂ profiles enhance the DCHP capacity by approximately +19.6% and +24%, respectively. However, this enhancement effect

generally diminishes as the foundation parameters k_1 and k_2 increase. For instance, when $k_2 = 0$ and k_1 increases from 1×10^6 to 3×10^6 , the influence of the FG_2 profile on the DCHP compared to the ZrO_2 shell decreases from 23.8% to 23.3%. Similarly, the effect of the FG_1 profile compared to the Si_3N_4 shell reduces from 19.6% to 19.3%. Similarly, when $k_1 = 1 \times 10^6$ and k_2 increases from 5×10^4 to 7×10^4 , the contribution of the FG_2 profile to the DCHP, compared to the ZrO_2 cylindrical shell, decreases from 22.9% to 22.6%, while that of the FG_1 profile, compared to the Si_3N_4 shell, reduces from 19.1% to 19%. Moreover, when $k_1 = 2 \times 10^6$ and k_2 increases from 5×10^4 to 7×10^4 , the enhancement effect of the FG_2 scheme drops from 22.6% to 22.4%, whereas the enhancement provided by the FG_1 profile decreases from 19% to 18.9%, relative to the corresponding homogeneous ceramic shells. Finally, for $k_1 = 3 \times 10^6$ and an increase in k_2 from 5×10^4 to 7×10^4 , the DCHP improvement of the FG_2 profile declines from 22.5% to 22.2%, and that of the FG_1 profile decreases from 19% to 18.8%.

Table 1. Variation of the DCHP values with respect to k_1 and k_2 for cylindrical shells made of Si_3N_4 , FG_1 , SUS304, ZrO_2 , FG_2 , and Ti-6Al-4V

$P_{1wp}^{Hcr} \times 10^5 (n_{cr})(n)$							
k_1	k_2	Si_3N_4	FG_1 -quad	SUS304	ZrO_2	FG_2 -quad	Ti-6Al-4V
0	0	1.116 (6)	0.897 (6)	0.747 (6)	1.146 (6)	0.871 (6)	0.721 (6)
1×10^6	0	1.123 (6)	0.903 (6)	0.753 (6)	1.158 (6)	0.883 (6)	0.733 (6)
	5×10^4	1.144 (6)	0.925 (6)	0.775 (6)	1.2 (6)	0.925 (6)	0.774 (6)
	7×10^4	1.153 (6)	0.934 (6)	0.783 (6)	1.216 (6)	0.941 (6)	0.791 (6)
2×10^6	0	1.129 (6)	0.91 (6)	0.76 (6)	1.17 (6)	0.895 (6)	0.745 (6)
	5×10^4	1.15 (6)	0.931 (6)	0.781 (6)	1.211 (6)	0.937 (6)	0.786 (6)
	7×10^4	1.159 (6)	0.94 (6)	0.79 (6)	1.228 (6)	0.953 (6)	0.803 (6)
3×10^6	0	1.135 (6)	0.916 (6)	0.766 (6)	1.182 (6)	0.907 (6)	0.756 (6)
	5×10^4	1.157 (6)	0.937 (6)	0.787 (6)	1.223 (6)	0.948 (6)	0.798 (6)
	7×10^4	1.165 (6)	0.946 (6)	0.796 (6)	1.24 (6)	0.965 (6)	0.814 (6)

When comparing the critical hydrostatic pressure values of cylindrical shells with FG_1 and FG_2 profiles to those of homogeneous Ti-6Al-4V and SUS304 shells, respectively, it is observed that, in the absence of a foundation, the FG_1 and FG_2 profiles result in reductions of approximately (-20.8%) and (-20.1%), respectively. This reduction effect generally weakens as the foundation stiffness parameters k_1 and k_2 increase. For example, when $k_2 = 0$ and k_1 increases from 1×10^6 to 3×10^6 , the reduction in the critical hydrostatic pressure for the FG_2 profile (compared to the Ti-6Al-4V shell) decreases from (-20.5%) to (-20%), while for the FG_1 profile (compared to the SUS304 shell), it decreases from (-19.9%) to (-19.6%).

Similarly, when $k_1 = 1 \times 10^6$ and k_2 increases from 5×10^4 to 7×10^4 , the reduction in critical hydrostatic pressure for the FG_2 model decreases from (-19.5%) to (-19%), and for the FG_1 profile from (-19.4%) to (-19.3%), in comparison to their corresponding homogeneous metallic shells. Moreover, when $k_1 = 2 \times 10^6$ and k_2 increases from 5×10^4 to 7×10^4 , the FG_2 scheme's reduction effect decreases from (-19.2%) to (-19%), while that of the FG_1 profile decreases from (-19.2%) to (-18.7%). Finally, for $k_1 = 3 \times 10^6$ and as k_2 increases from 5×10^4 to 7×10^4 , the critical hydrostatic

pressure reduction for the FG₂ profile declines from (–16.5%) to (–16.1%), and for the FG₁ profile from (–19.1%) to (–18.8%), compared to Ti-6Al-4V and SUS304 shells, respectively.

Fig. 2 presents the variation of the DCHP as a function of the material grading index N for cylindrical shells with FG₁ and FG₂ profiles, both with and without elastic foundation support. The geometric parameters are taken as $r/t=100$ ve $a/r=2$, while the foundation stiffness coefficients are selected as $k_1=k_2=0$, $k_1=2\times10^6, k_2=0$, and $k_1=2\times10^6, k_2=5\times10^4$. It is observed that increasing the value of N leads to a decrease in DCHP values for both FG₁ and FG₂ profiles. When comparing the FG₁ and FG₂ profiles, the minimum DCHP values occur at $N=3$, while the maximum values are obtained at $N=0.5$. The most pronounced influence of the Pasternak foundation occurs in the FG₂ shell with $N=2$, where the effect reaches 7.6%. In contrast, the weakest foundation influence is observed in the presence of the Winkler foundation, yielding only 1.2% for the FG₁ shell with $N=0.5$. The increase in the power-law index N leads to a regular and pronounced enhancement in the influence of the Pasternak foundation on the critical hydrostatic load, particularly for $N\leq 2$. Although the effect continues to grow beyond $N=2$, the rate of increase becomes noticeably weaker. In contrast, the influence of the Winkler foundation, while generally increasing, exhibits an irregular pattern and does not follow a consistent trend. As the N exponent increases from 0.5 to 3, the FGM effect on DCHP increases.

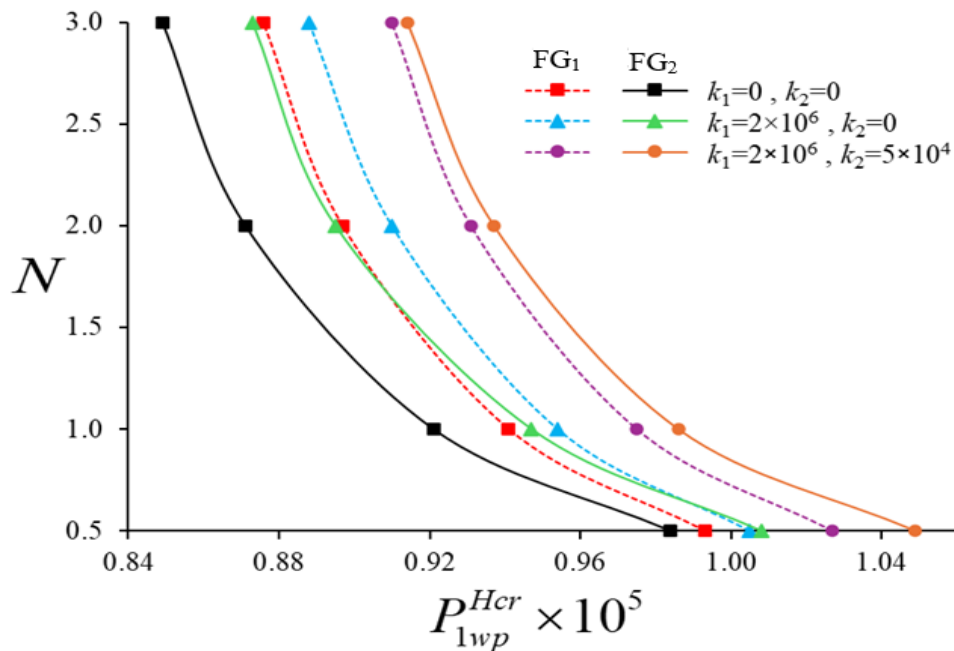


Fig. 2. Variation of the DCHP with respect to the material grading index N for cylindrical shells with FG₁ and FG₂ profiles, with and without elastic foundation support

5. Conclusions. In this investigation, the stability behavior of functionally graded cylindrical shells subjected to external hydrostatic loading is examined within the framework of Donnell-type shell theory resting on an elastic foundation. The study first formulates the governing relations describing the material gradation, elastic foundation response, and hydrostatic loading in a unified mathematical setting. Thereafter, employing a combined Pasternak–Winkler foundation model, the corresponding stability and compatibility equations are rigorously derived based on the classical Kirchhoff–Love shell assumptions. To facilitate an analytical treatment, admissible mode shapes satisfying fixed–fixed boundary conditions are introduced, and closed-form solutions are obtained through the application of the Galerkin procedure. A comprehensive parametric study is then carried out by varying the material distribution and foundation stiffness parameters in order to quantify their individual and combined effects. The results of the numerical investigation yield several overarching conclusions:

- a) The FG_2 gradation consistently provides enhanced stability characteristics, particularly under conditions involving large foundation stiffness.
- b) Although the FG_1 profile exhibits a comparatively steeper reduction in critical load when the foundation stiffness is negligible or small, the presence of an elastic foundation significantly narrows this performance gap.
- c) For design scenarios where high stability margins are essential, the FG_2 configuration and ceramic-rich compositions are recommended, whereas the FG_1 profile offers advantages in applications prioritizing reduced weight alongside moderate stability requirements.

Collectively, these findings underscore that an informed selection of material gradation patterns and foundation parameters plays a decisive role in the development of next-generation structural components, enabling designers to achieve an optimal balance among mechanical robustness, lightweight performance, and application-specific functional criteria.

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