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HIGHER-ORDER THEORIES FOR DOUBLY-CURVED LAMINATED ANISOTROPIC SHELL STRUCTURES WITH GENERALIZED DIFFERENTIAL QUADRATURE METHOD

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Abstract. A refined two-dimensional (2D) formulation is presented for evaluating the static and free vibrational response of laminated anisotropic doubly-curved shell structures. The model is based on a unified formulation and accounts for a kinematic model which uses higher-order polynomials. The governing equations are derived from the Hamilton principle in principal coordinates, explicitly accounting for curvature effects and arbitrarily shaped external surface loads. A numerical solution is obtained by approximating the strong form governing equations with the Generalized Differential Quadrature (GDQ) method. In the post-processing, the strain and stress distributions of the corresponding three-dimensional (3D) solid are reconstructed through an equilibrium-based recovery procedure. Some numerical examples are provided to validate the model against reference 3D finite-element solutions obtained with commercial software. In addition, the influence of the through-the-thickness kinematic model on the accuracy of the results is assessed for both natural frequency prediction and static response of the laminate. The proposed approach provides an efficient and reliable tool for analyzing doubly-curved laminated structures made of advanced materials, offering high accuracy with reduced computational cost.

Keywords: anisotropic materials, doubly-curved shells, equivalent single layer, generalized differential quadrature

1.Introduction. Novel applications across engineering fields increasingly demand advanced structural models capable of accurately predicting the response of components made from innovative materials and featuring complex geometries, all with reduced computational cost. In the context of laminated structures, two-dimensional (2D) formulations are widely adopted and typically classified into Equivalent Single Layer (ESL) and Layer-Wise (LW) theories. ESL theories reduce the three-dimensional (3D) mechanics of a laminated panel to that of a reference surface located at the mid-thickness of the structure [1], whereas LW theories assign a reference surface to each layer. Consequently, the geometric description of the reference surface becomes a key element of LW models, requiring the use of differential geometry principles—particularly when parametrization in curvilinear principal coordinates is available [2]. In the absence of such parametrization, a mapping of the physical domain must be introduced. Another critical aspect related to 2D formulations is the kinematic model assumed along the thickness direction. Classical approaches, such as Kirchhoff-Love Plate Theory (CPT) and the First-order Shear Deformation Theory (FSDT), assume linear through-the-thickness distributions of the in-plane displacement field variables, while assigning a uniform distribution to the out-of-plane displacement. To improve the accuracy, the Third-order Shear Deformation Theory (TSDT) introduces a third-order polynomial for in-plane displacements. To overcome the limitations of classical approaches, higher-order theories have been introduced, which employ higher-order polynomials to describe the unknown field variables. Within this framework, the unified formulation has emerged as an effective modeling strategy, introducing generalized thickness functions to construct flexible kinematic models. These higher-order 2D formulations enable more refined predictions than

classical models. Analytical solutions of the governing equations are possible for selected geometries, materials, loads, and boundary conditions, providing efficient benchmarks [3]. However, for most engineering applications, closed-form solutions are not feasible, requiring the adoption of numerical techniques. The Finite Element Method (FEM) remains the most widely used approach but may become computationally expensive, particularly for components with geometric discontinuities or complex boundaries. As an efficient alternative, the Generalized Differential Quadrature (GDQ) method directly discretizes the strong form of the governing equations using a global interpolation of derivatives [4]. This method has proven highly accurate across various engineering problems, achieving excellent agreement with reference solutions while drastically reducing the number of Degrees of Freedom (DOFs) compared to FEM.

In this work, the accuracy of higher-order theories is assessed for the analysis of anisotropic doubly-curved shell structures. The governing equations are derived from the Hamilton principle, considering the elastic strain energy, kinetic energy, and the virtual work of external loads. The kinematic model is based on the unified formulation and incorporates zigzag functions to predict the interaction between adjacent layers. A NURBS-based mapping of the physical domain is introduced to handle arbitrarily-shaped shell geometries. Numerical solutions are obtained with the GDQ method, while post-processing involves reconstructing through-thickness strain and stress distributions using a recovery procedure [5] based on 3D equilibrium equations. Natural frequencies are also determined. The validation is carried out by comparing the GDQ predictions against 3D FEM reference solutions [6] obtained from commercial software, confirming the model accuracy for both static and free vibration analyses. The influence of the kinematic expansion order is further examined. Overall, the proposed formulation provides an efficient and reliable tool for analyzing laminated structural components of complex shape, combining the accuracy of higher-order theories with the computational efficiency of GDQ.

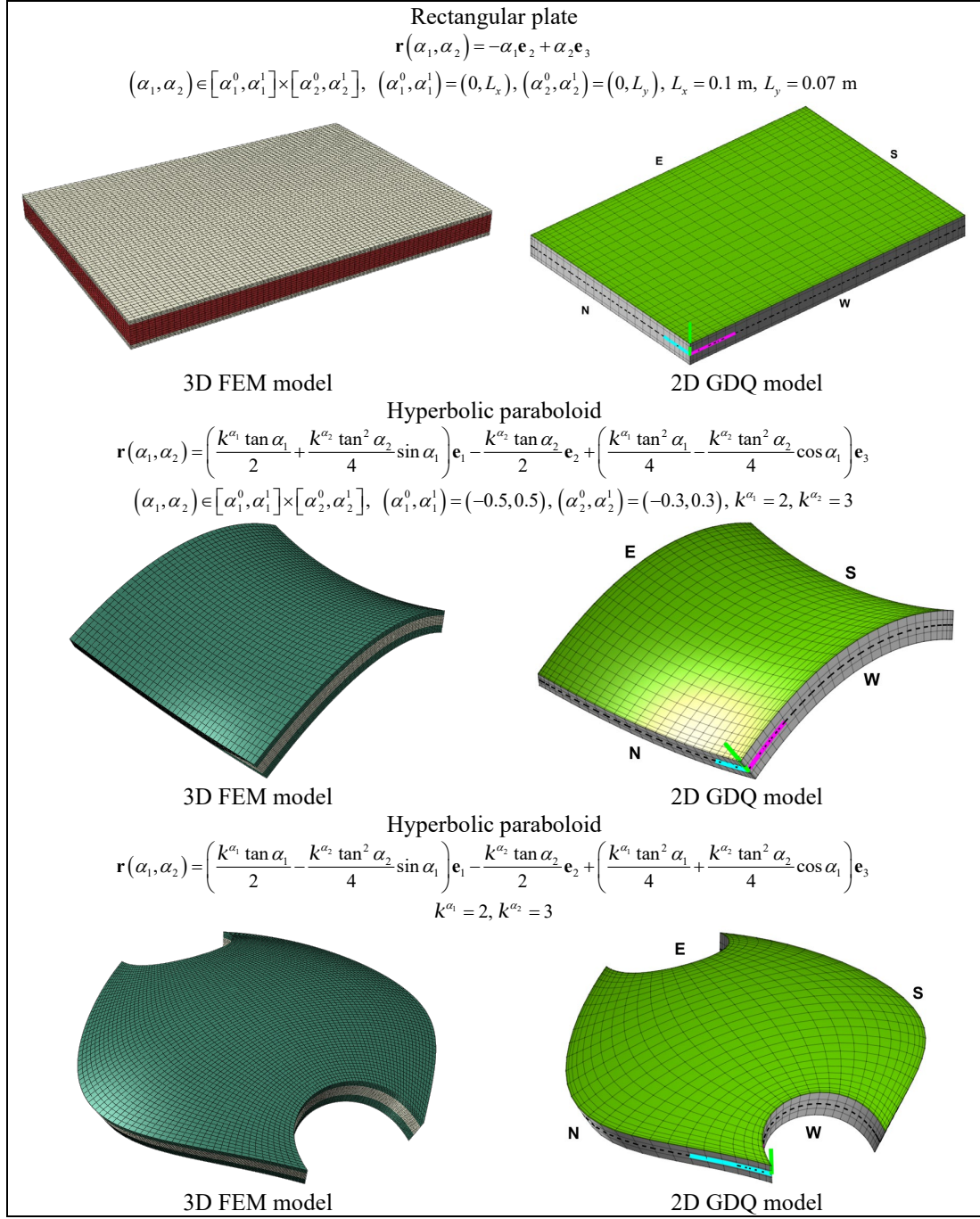


Figure 1. Position vector and representation of the laminated structures considered in the simulation.

Theoretical formulation. In this section, the theoretical framework for the adoption of higher-order theories for the static and dynamic analysis of laminated shell structures is briefly outlined. The formulation begins with the geometric description of the structure in an appropriate system of curvilinear principal coordinates. Within this setting, the main geometric features of the reference surface, represented by its position vector $\mathbf{r}(\alpha_1, \alpha_2)$, are derived using the main principles of differential geometry. These quantities include the Lamé parameters A_1, A_2 , and the principal radii of curvature R_1, R_2 , among others. An arbitrary point of the doubly-curved 3D solid can, thus, be expressed in terms of the position vector $\mathbf{R}$, defined as:

$$\mathbf{R}(\alpha_1, \alpha_2, \zeta) = \mathbf{r}(\alpha_1, \alpha_2) + \frac{h(\alpha_1, \alpha_2)}{2} \mathbf{z} \mathbf{n}(\alpha_1, \alpha_2) \quad (1)$$

In the previous equation, $h(\alpha_1, \alpha_2)$ denotes the total thickness of the structure, obtained as the sum of the thicknesses h_k with $k=1, \dots, l$ of each layer in the lamination scheme, while $z = 2\zeta/h$ is a dimensionless thickness coordinate. The vector $\mathbf{n}(\alpha_1, \alpha_2)$ represents the outward normal unit vector to the reference surface. The geometric coordinates in Eq. (1) are, therefore, the principal directions α_1, α_2 of the reference surface, which vary within the rectangular interval $[\alpha_1^0, \alpha_1^1] \times [\alpha_2^0, \alpha_2^1]$, together with the thickness coordinate ζ , with $\zeta \in [-h/2, h/2]$. In case of arbitrarily-shaped structures that cannot be described using principal coordinates, a generalized mapping procedure is employed. This approach is based on blending functions that use NURBS curves to represent the arbitrary geometry. Accordingly, the natural coordinates ξ_1, ξ_2 are introduced, which vary within a dimensionless parent element $[-1, 1]$.

$$\begin{aligned} \alpha_1(\xi_1, \xi_2) &= \frac{1}{2} \left((1-\xi_2) \bar{\alpha}_{1(1)}(\xi_1) + (1+\xi_1) \bar{\alpha}_{1(2)}(\xi_2) + (1+\xi_2) \bar{\alpha}_{1(3)}(\xi_1) + (1-\xi_1) \bar{\alpha}_{1(4)}(\xi_2) \right) + \\ &\quad - \frac{1}{4} \left((1-\xi_1)(1-\xi_2) \alpha_{1(1)} + (1+\xi_1)(1-\xi_2) \alpha_{1(2)} + (1+\xi_1)(1+\xi_2) \alpha_{1(3)} + (1-\xi_1)(1+\xi_2) \alpha_{1(4)} \right) \\ \alpha_2(\xi_1, \xi_2) &= \frac{1}{2} \left((1-\xi_2) \bar{\alpha}_{2(1)}(\xi_1) + (1+\xi_1) \bar{\alpha}_{2(2)}(\xi_2) + (1+\xi_2) \bar{\alpha}_{2(3)}(\xi_1) + (1-\xi_1) \bar{\alpha}_{2(4)}(\xi_2) \right) + \\ &\quad - \frac{1}{4} \left((1-\xi_1)(1-\xi_2) \alpha_{2(1)} + (1+\xi_1)(1-\xi_2) \alpha_{2(2)} + (1+\xi_1)(1+\xi_2) \alpha_{2(3)} + (1-\xi_1)(1+\xi_2) \alpha_{2(4)} \right) \end{aligned} \quad (2)$$

In line with the ESL modeling of laminated structures, the displacement field variables, arranged in the vector $\mathbf{U}^{(k)} = [U_1^{(k)} \ U_2^{(k)} \ U_3^{(k)}]^T$, are expanded in terms of generalized thickness functions $F^{(k\tau)\alpha_i}$ with $i=1, 2, 3$, defined for any $\tau=0, \dots, N+1$. In a compact form, this can be expressed as:

$$\mathbf{U}^{(k)}(\alpha_1, \alpha_2, \zeta, t) = \sum_{\tau=0}^{N+1} \mathbf{F}^{(k\tau)} \mathbf{u}^{(\tau)} \quad (3)$$

where $\mathbf{u}^{(\tau)} = [u_1^{(\tau)} \ u_2^{(\tau)} \ u_3^{(\tau)}]^T$ is the vector of the generalized displacement field components defined on the reference surface, while $\mathbf{F}^{(k\tau)}$ is the thickness function matrix, of size 3×3 , for any $\tau=0, \dots, N+1$. In this work, the thickness functions are constructed using power polynomials for $\tau=0, \dots, N$. In addition, for $\tau=N+1$, the Murakami's function is used, which enables the representation of the well-known zigzag effect:

$$F_t^{(k)\alpha_i}(\zeta) = \begin{cases} \zeta^\tau & \tau = 0, \dots, N \\ (-1)^k z_k = (-1)^k \frac{2}{\zeta_{k+1} - \zeta_k} \zeta - \frac{\zeta_{k+1} + \zeta_k}{\zeta_{k+1} - \zeta_k} & \tau = N+1 \end{cases} \quad (4)$$

Starting from the kinematic model introduced in Eq. (3), the strain-displacement are derived. By using higher-order theories, the 3D strain vector $\mathbf{\epsilon}^{(k)}(\alpha_1, \alpha_2, \zeta, t) = [\mathcal{E}_1^{(k)} \ \mathcal{E}_2^{(k)} \ \gamma_{12}^{(k)} \ \gamma_{13}^{(k)} \ \gamma_{23}^{(k)} \ \mathcal{E}_3^{(k)}]^T$ is expanded as follows, where $\mathbf{\epsilon}^{(\tau)\alpha_i} = [\mathcal{E}_1^{(\tau)\alpha_i} \ \mathcal{E}_2^{(\tau)\alpha_i} \ \gamma_{12}^{(\tau)\alpha_i} \ \gamma_{13}^{(\tau)\alpha_i} \ \gamma_{23}^{(\tau)\alpha_i} \ \mathcal{E}_3^{(\tau)\alpha_i}]^T$ is the vector of the generalized strain components:

$$\mathbf{\epsilon}^{(k)} = \sum_{\tau=0}^{N+1} \sum_{i=1}^3 \mathbf{Z}^{(k\tau)\alpha_i} \mathbf{\epsilon}^{(\tau)\alpha_i} \quad (5)$$

Additional details on the derivation of the generalized definition operator $\mathbf{Z}^{(k\tau)\alpha_i}$ in Eq. (5) can be found in Ref. [4]. The constitutive relationship adopted in this theory accounts for a generally anisotropic behavior at each point of the 3D solid, in line with the expression reported below:

$$\boldsymbol{\sigma}^{(k)} = \bar{\mathbf{E}}^{(k)} \mathbf{\epsilon}^{(k)} \leftrightarrow \begin{bmatrix} \sigma_1^{(k)} \\ \sigma_2^{(k)} \\ \tau_{12}^{(k)} \\ \tau_{13}^{(k)} \\ \tau_{23}^{(k)} \\ \sigma_3^{(k)} \end{bmatrix} = \begin{bmatrix} \bar{E}_{11}^{(k)} & \bar{E}_{12}^{(k)} & \bar{E}_{16}^{(k)} & \bar{E}_{14}^{(k)} & \bar{E}_{15}^{(k)} & \bar{E}_{13}^{(k)} \\ \bar{E}_{12}^{(k)} & \bar{E}_{22}^{(k)} & \bar{E}_{26}^{(k)} & \bar{E}_{24}^{(k)} & \bar{E}_{25}^{(k)} & \bar{E}_{23}^{(k)} \\ \bar{E}_{16}^{(k)} & \bar{E}_{26}^{(k)} & \bar{E}_{66}^{(k)} & \bar{E}_{46}^{(k)} & \bar{E}_{56}^{(k)} & \bar{E}_{36}^{(k)} \\ \bar{E}_{14}^{(k)} & \bar{E}_{24}^{(k)} & \bar{E}_{46}^{(k)} & \bar{E}_{44}^{(k)} & \bar{E}_{45}^{(k)} & \bar{E}_{34}^{(k)} \\ \bar{E}_{15}^{(k)} & \bar{E}_{25}^{(k)} & \bar{E}_{56}^{(k)} & \bar{E}_{45}^{(k)} & \bar{E}_{55}^{(k)} & \bar{E}_{35}^{(k)} \\ \bar{E}_{13}^{(k)} & \bar{E}_{23}^{(k)} & \bar{E}_{36}^{(k)} & \bar{E}_{34}^{(k)} & \bar{E}_{35}^{(k)} & \bar{E}_{33}^{(k)} \end{bmatrix} \begin{bmatrix} \mathcal{E}_1^{(k)} \\ \mathcal{E}_2^{(k)} \\ \gamma_{12}^{(k)} \\ \gamma_{13}^{(k)} \\ \gamma_{23}^{(k)} \\ \mathcal{E}_3^{(k)} \end{bmatrix} \quad (6)$$

with $k=1, \dots, l$. In the previous equation, $\boldsymbol{\sigma}^{(k)}$ denotes the vector of 3D stress components, while $\bar{\mathbf{E}}^{(k)}$ represents the 3D stiffness matrix expressed in the geometric reference system. This matrix is obtained by rotating the stiffness matrix of each individual layer, $\mathbf{E}^{(k)}$, from the material reference system to the geometric reference system through an angle $\mathcal{g}^{(k)}$. To this end, the rotation matrix $\mathbf{T}^{(k)} = \mathbf{T}^{(k)}(\mathcal{g}^{(k)})$ is conveniently introduced. It gives:

$$\bar{\mathbf{E}}^{(k)} = \mathbf{T}^{(k)} \mathbf{E}^{(k)} (\mathbf{T}^{(k)})^T \quad (7)$$

From the computation of the virtual variation of the elastic strain energy, generalized stress components are defined for each τ -th order of the kinematic expansion. These quantities are grouped into the vector $\mathbf{S}^{(\tau)\alpha_i} = [N_1^{(\tau)\alpha_i} \ N_2^{(\tau)\alpha_i} \ N_{12}^{(\tau)\alpha_i} \ N_{21}^{(\tau)\alpha_i} \ T_1^{(\tau)\alpha_i} \ T_2^{(\tau)\alpha_i} \ P_1^{(\tau)\alpha_i} \ P_2^{(\tau)\alpha_i} \ S_3^{(\tau)\alpha_i}]^T$. As a consequence, the 3D constitutive relation in Eq. (6) can be expressed in the ESL formulation as follows:

$$\mathbf{S}^{(\tau)\alpha_i} = \sum_{\eta=0}^{N+1} \sum_{j=1}^3 \mathbf{A}^{(\tau\eta)\alpha_i\alpha_j} \mathbf{\epsilon}^{(\eta)\alpha_j} \quad (8)$$

being $\mathbf{\epsilon}^{(\eta)\alpha_j}$ the higher-order strain vector of Eq. (5), while $\mathbf{A}^{(\tau\eta)\alpha_i\alpha_j}$ denotes the generalized stiffness matrix. Additional details can be found in Ref. [4].

The governing equations are derived from the Hamilton principle by evaluating the elastic strain energy of the 3D solid, the kinetic energy, and the virtual work of external surface loads. By applying the integration by parts rule, the higher-order equilibrium equations are obtained for any τ -th kinematic expansion order, along with the corresponding boundary conditions on the four edges of the 2D physical domain. Then, by introducing the generalized constitutive relation of Eq. (8) the definition equations of Eq. (5), and the kinematic model of Eq. (3), the fundamental

equations are formulated in terms of the generalized displacement field components. These equations can be expressed in compact matrix form in the relation reported below for any $\tau = 0, \dots, N+1$:

$$\sum_{\eta=0}^{N+1} \mathbf{L}^{(\tau\eta)} \mathbf{u}^{(\eta)} - \sum_{\eta=0}^{N+1} \mathbf{M}^{(\tau\eta)} \ddot{\mathbf{u}}^{(\eta)} + \mathbf{q}^{(\tau)} = \mathbf{0} \quad (9)$$

In the previous equation, $\mathbf{L}^{(\tau\eta)}$ denotes the fundamental operator, which incorporates both the equilibrium equations and the definition equations, while $\mathbf{M}^{(\tau\eta)}$ represents the higher-order mass matrix. Finally, the vector $\mathbf{q}^{(\tau)} = [q_1^{(\tau)} \quad q_2^{(\tau)} \quad q_3^{(\tau)}]^T$ contains the generalized surface loads applied on the reference surface, derived from the static equivalence principle by evaluating the virtual work of the loads applied to the top and bottom surfaces of the shell. The complete expressions of these quantities can be found in Ref. [4].

A closed-form analytical solution of the fundamental relations of Eq. (9) can be derived under specific assumptions on geometry, material properties, loading, and boundary conditions, as discussed in Ref. [4]. For more general cases, the solution is computed numerically using the GDQ method. According to the GDQ rule, if $f = f(x)$ is an arbitrary smooth function defined over the closed interval $x \in [a, b]$, the n -th order derivative of f at a discrete point x_i with $i = 1, \dots, I_Q$ is approximated using the following quadrature formula, where $\zeta_{ij}^{(n)}$ are the weighted coefficients, which are calculated recursively [4]:

$$f^{(n)}(x_i) = \left. \frac{\partial^n f(x)}{\partial x^n} \right|_{x=x_i} \cong \sum_{j=1}^{I_Q} \zeta_{ij}^{(n)} f(x_j) \Leftrightarrow \mathbf{f}^{(n)} = \boldsymbol{\zeta}^{(n)} \mathbf{f} \quad (10)$$

The discrete points used in Eq. (10) follow the well-known Chebyshev-Gauss-Lobatto (CGL) distribution, as discussed in detail in Ref. [4]. As a consequence, the size of the discrete vectors $\mathbf{f}^{(n)}$ and $\mathbf{f}$ is $I_Q \times 1$. The matrix form of Eq. (10) also facilitates the extension of the one-dimensional GDQ rule explained above to the 2D case through the use of the Kronecker product.

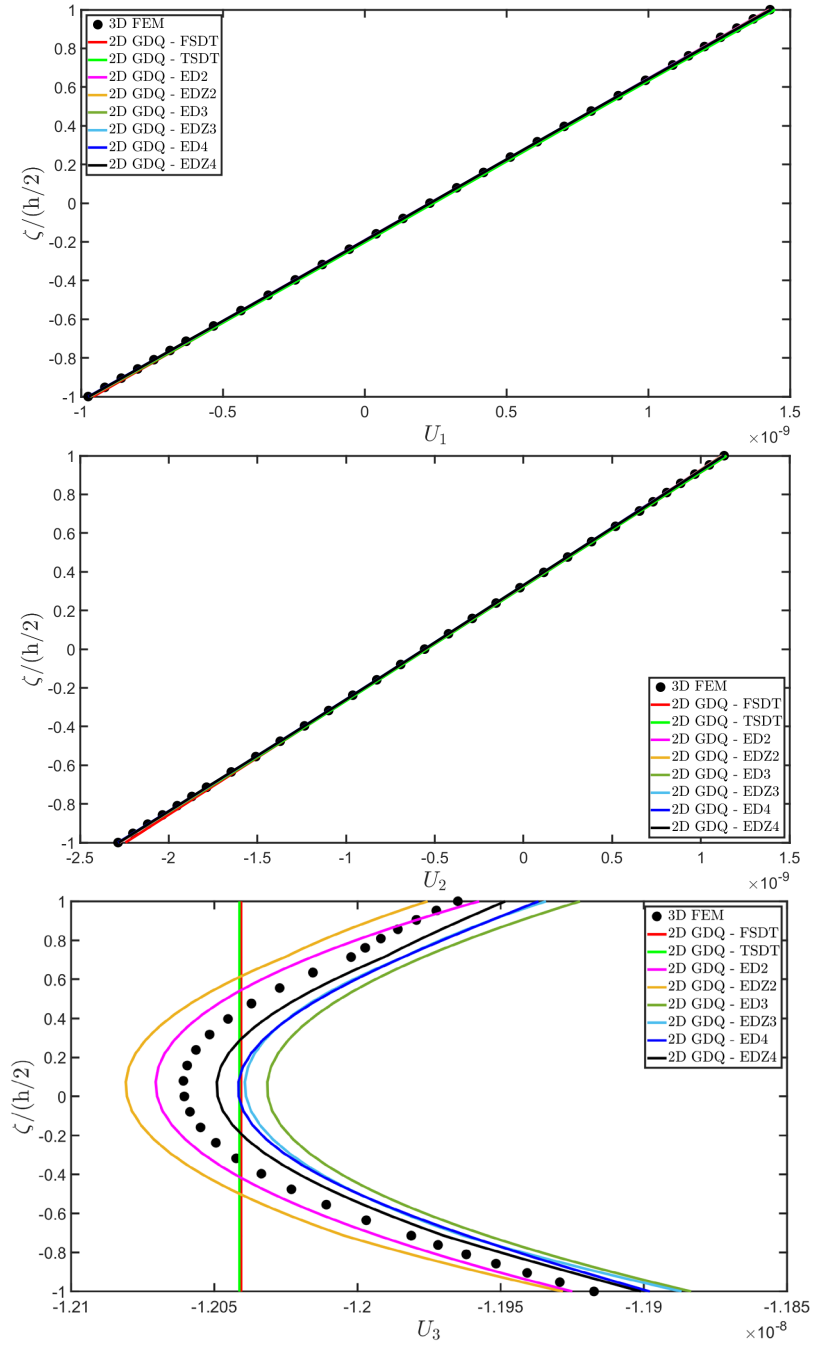


Figure 2. Through-the-thickness distribution of the 3D displacement field components, expressed in m and evaluated at $(0.25L_1, 0.25L_2)$ within the physical domain of a simply-supported rectangular plate made of three layers of generally anisotropic materials subjected to an external pressure and lying on an elastic foundation.

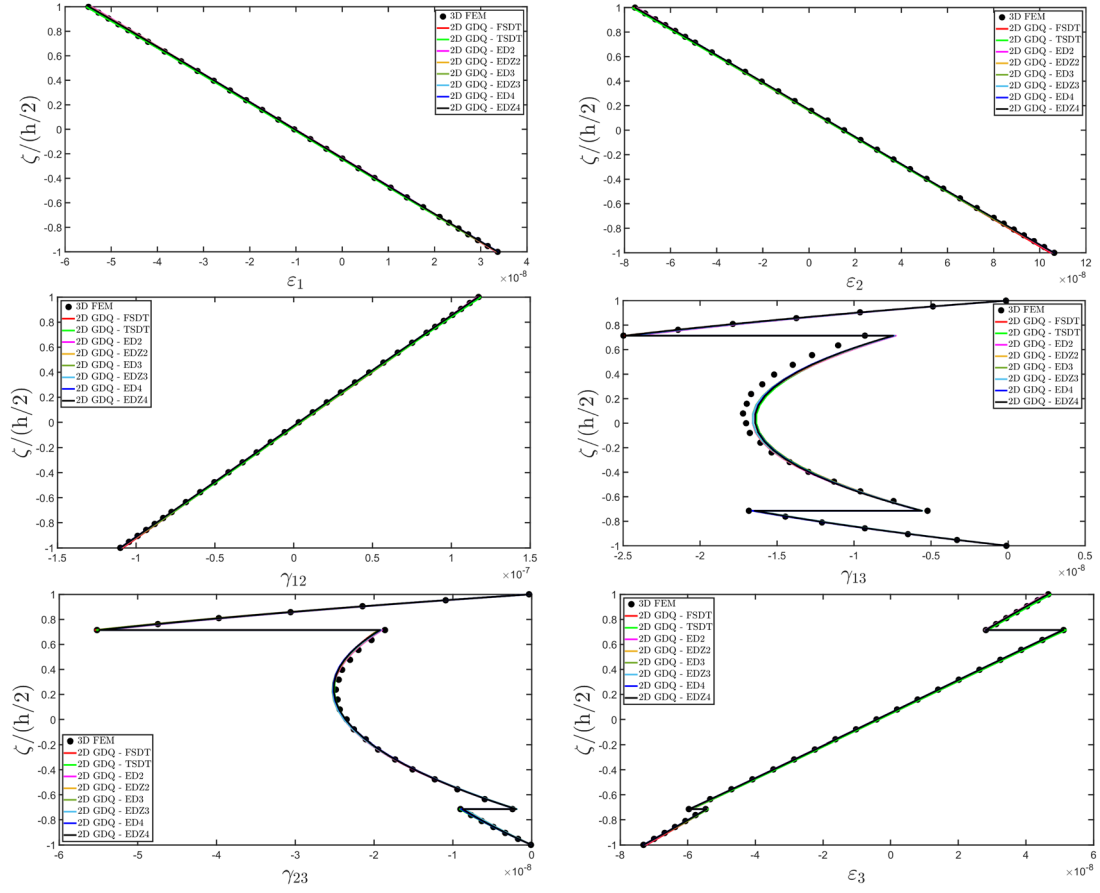


Figure 3. Through-the-thickness distribution of the 3D strain components, expressed in [m/m] and evaluated at $(0.25L_1, 0.25L_2)$ within the physical domain of a simply-supported rectangular plate made of three layers of generally anisotropic materials subjected to an external pressure and lying on an elastic foundation.

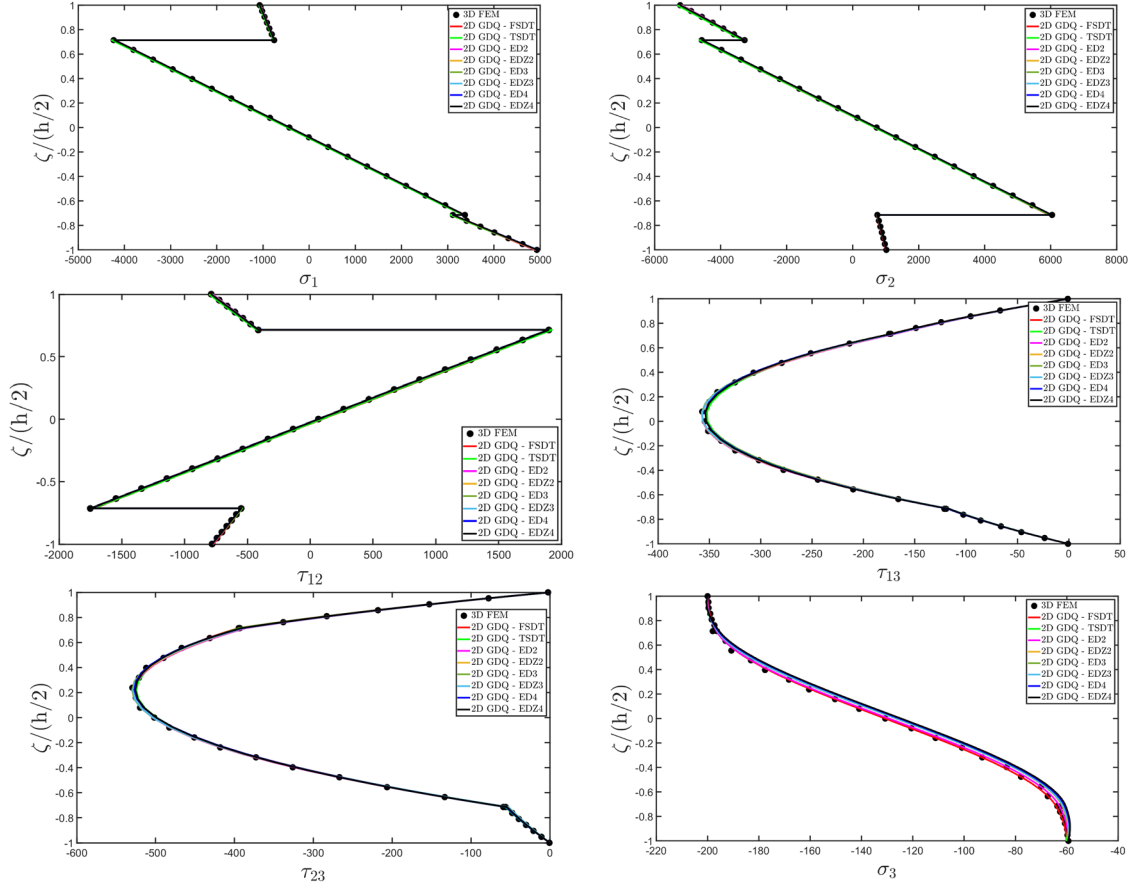


Figure 4. Through-the-thickness distribution of the 3D stress components, expressed in $[\text{N/m}^2]$ and evaluated at $(0.25L_1, 0.25L_2)$ within the physical domain of a simply-supported rectangular plate made of three layers of generally anisotropic materials subjected to an external pressure and lying on an elastic foundation.

Applications and results. In this section, some numerical investigations are presented to assess the structural response of anisotropic panels with different curvatures using the 2D higher-order shell theory. The analysis includes the evaluation of stress and strain distributions, natural frequencies, and mode shapes. For each case, the same analyses are performed using a refined 3D FEM model from commercial software to provide a reference solution. In particular, the panels considered in these investigations are illustrated in Fig. 1. For each panel, the figure provides the reference surface equation for a rectangular plate, an elliptic paraboloid, and a hyperbolic paraboloid, along with a representation of the 3D FEM mesh, which employs parabolic brick elements named C3D20. The 2D GDQ solution is obtained using the well-known CGL distribution, considering a 2D grid made of $I_N \times I_M = 31 \times 31$ sample points. Each structure consists

of three-layers. In particular, the central core is made of triclinic material, and the external laminae are made of graphite-epoxy. The mechanical properties of these materials are taken from Ref. [1].

The first example considers a simply-supported rectangular plate, where the thickness of each lamina is $h_1 = h_3 = 0.001\text{ m}$ and $h_2 = 0.005\text{ m}$. The lamination scheme is $(0/20/70)$. The panel rests on a Winkler elastic foundation, characterized by linear elastic springs oriented along the thickness direction with stiffness $k_{3f}^{(-)} = 5 \times 10^9 \text{ N/m}^3$. A sinusoidally-distributed mechanical pressure of magnitude $\bar{q}_3^{(+)} = 4 \times 10^2 \text{ N/m}^2$ is applied on the top surface, described by the relation reported below:

$$q_3^{(+)}(x, y) = \bar{q}_3^{(+)} \sin\left(\frac{x}{L_1}\right) \sin\left(\frac{y}{L_2}\right) \quad (11)$$

The results are presented in terms of through-the-thickness distributions of the displacement field components, 3D strains, and 3D stress at points located at $(0.25L_1, 0.25L_2)$ within the 2D physical domain, as shown in Figs. 2-4. For each quantity, a 3D FEM solution is provided, as obtained from a numerical model from commercial software, consisting of 65280 elements and 280411 nodes, for a total number of 841233 DOFs. As observed in Fig. 2, the in-plane displacement components exhibit linear distributions, while the out-of-plane component shows a parabolic profile. As a consequence, classical approaches such as FSDT and TSDT cannot be adopted to reproduce results consistent with the 3D FEM solution. On the other hand, the adoption of a higher-order kinematic model significantly improves the accuracy. Increasing the maximum order of the kinematic expansion translates the curve closer to the reference solution, yielding better agreement with the 3D FEM results. On the other hand, the 3D strain components of Fig. 3 predicted by the higher-order model closely match those from the reference solution. The same trend is observed for the 3D stress components in Fig. 4. In particular, the adoption of a higher-order kinematic model, as in the EDZ4 theory, allows for a more accurate evaluation of the out-of-plane stress components through the use of the recovery procedure.

The theory is then applied to determine the natural frequencies and mode shapes of doubly-curved shell structures. In this example, a hyperbolic paraboloid is considered, defined over the rectangular physical domain $(\alpha_1^0, \alpha_1^1) \times (\alpha_2^0, \alpha_2^1) = (-0.5, 0.5) \times (-0.3, 0.3)$, as can be seen from Fig. 1. The lamination scheme consists of two external layers of graphite-epoxy and a central core of triclinic material, oriented according to the stacking sequence $(20/35/75)$. The thicknesses of the three laminae are $h_1 = 0.03\text{ m}$, $h_2 = 0.04\text{ m}$ and $h_3 = 0.02\text{ m}$. The first ten natural frequencies of the panel under consideration are evaluated under CFFF boundary conditions, where C denotes a clamped edge ($u_1^{(r)} = u_2^{(r)} = u_3^{(r)} = 0$), while F denotes the free boundary condition, corresponding to null generalized stress resultants $s^{(r)\alpha_i}$ along the remaining edges [4]. More specifically, the structure is clamped along the lateral surface corresponding to the edge of the physical domain characterized by $\alpha_2 = \alpha_2^0$. Natural frequencies are reported in Table 1. Here, a successful agreement is shown with the results obtained from a 3D FEM model consisting of 156952 nodes and 36450 elements. In particular, the results demonstrate that the use of a higher-order kinematic model, combined with the zigzag function, is essential to accurately predict the 3D vibrational response of the structure. Finally, Fig. 5 presents a 3D representation of the first nine mode shapes of the structure, obtained from the 2D model employing the EDZ4 kinematic field.

The final numerical example evaluates the accuracy of the theory for arbitrarily-shaped structures. An elliptic paraboloid is considered, with its physical domain distorted using the blending functions of Eq. (2), where the four edges of the mapped physical domain are described with the NURBS curves adopted in Ref. [1]. The lamination scheme is the same as the previous example, while the boundary conditions are FCFF, meaning that the edge associated with $\xi_1 = \xi_1^0$ in the parent domain is clamped, whereas the remaining edges are free. The first ten natural frequencies are reported in Table 2 for various higher-order theories. The reference solution is obtained from a 3D FEM model developed with commercial software, made of C3D20 parabolic brick elements and 720756 DOFs. Also in this case, the results demonstrate that the refined 2D theory accurately predicts the response of the reference solution, while requiring significantly less computational effort. The accuracy is particularly evident in those simulations that consider higher-order thickness functions and the zigzag function. Finally, the first nine mode shapes, obtained from the EDZ4 simulation, are presented in Fig. 6.

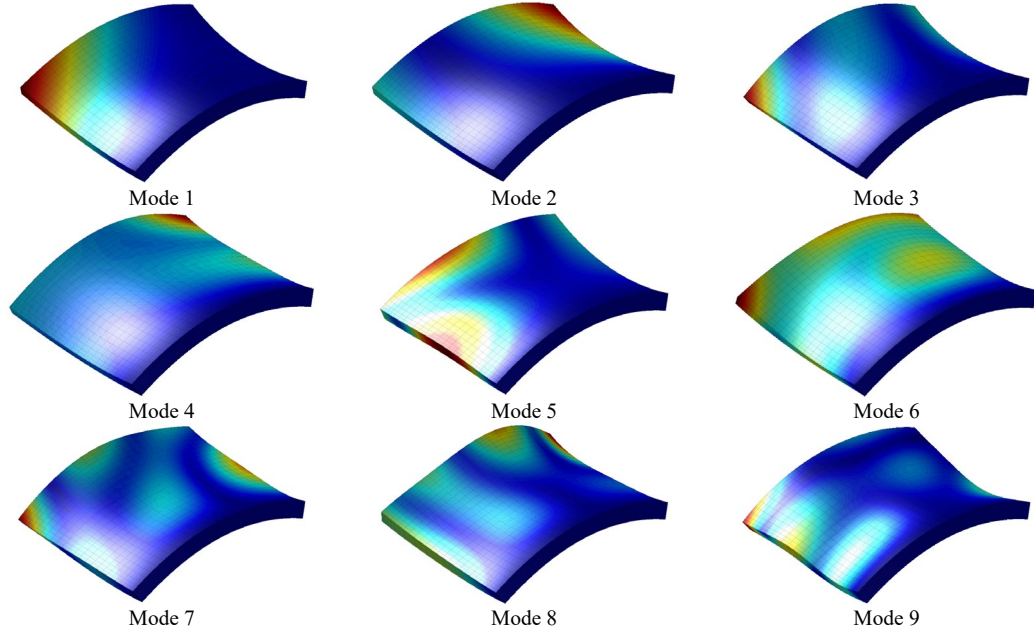


Figure 5. Three-dimensional representation of the first nine mode shapes of a laminated anisotropic hyperbolic paraboloid described with principal coordinates.

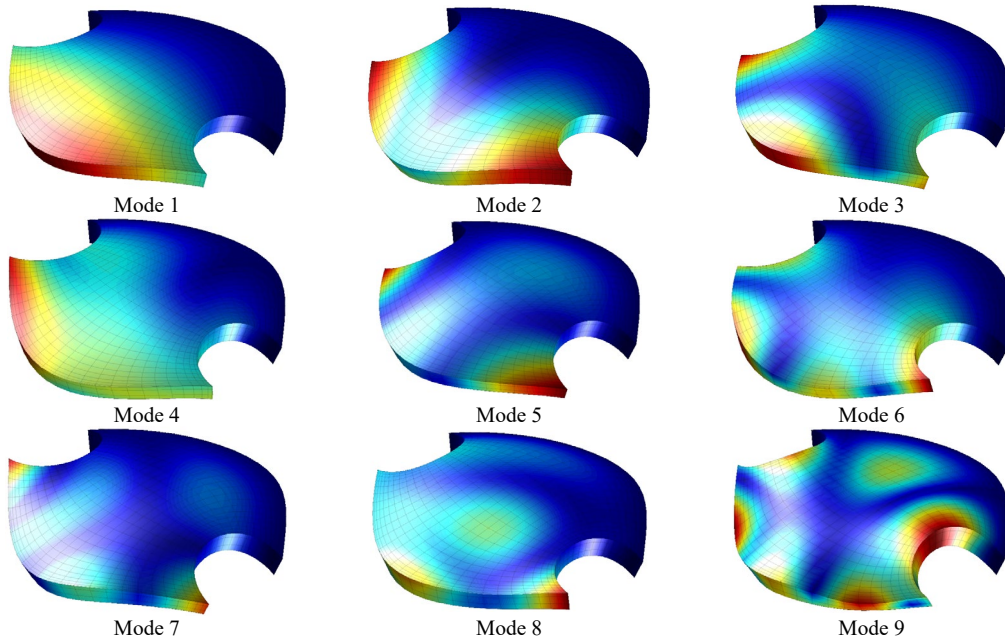


Figure 6. Three-dimensional representation of the first nine mode shapes of a laminated anisotropic elliptic paraboloid described with arbitrarily-shaped domains.

Conclusions. A higher-order ESL model has been presented for the static and dynamic analysis of anisotropic doubly-curved shell structures with arbitrary three-dimensional variations in the material orientation. The formulation is based on a generalized expansion of the unknown field variables, considering both through-the-thickness stretching and zigzag effects between adjacent laminae. The governing equations are derived from the Hamilton principle and solved numerically using the GDQ method. The theory has been successfully applied to predict the static response and natural frequencies of both flat and curved laminated panels with various lamination schemes, showing high accuracy compared to computationally expensive simulations. Moreover, it is shown that this model facilitates the analysis of complex structures that would otherwise require advanced modelling skills when using standard commercial software.

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Table 1. First ten natural frequencies of a hyperbolic paraboloid described in curvilinear principal coordinates employing higher-order theories. Effect of the kinematic model on the accuracy of the solution.

f [Hz]	3D FEM	FSDT	TSDT	EDZ1	ED2	EDZ2	ED3	EDZ3	ED4	EDZ4
DOFs	470856	5046	10092	7569	7569	10092	10092	12615	12615	15138
1	77.20	77.07	77.13	81.35	77.28	77.22	77.49	77.38	77.44	77.33
2	105.91	105.94	106.06	107.68	106.04	105.96	106.35	106.15	106.29	106.09
3	271.14	271.39	271.42	287.17	271.36	271.04	272.41	271.80	272.21	271.54
4	351.81	351.87	352.85	359.09	352.15	351.60	354.23	353.00	354.00	352.58
5	371.33	372.13	372.64	378.77	371.58	371.07	373.12	372.14	372.89	371.77
6	464.62	465.23	465.85	474.82	464.95	464.55	466.38	465.49	466.18	465.18
7	523.07	524.43	526.27	549.22	523.19	522.47	526.93	524.84	526.46	524.35
8	637.18	643.55	644.08	659.28	637.91	636.41	644.17	639.74	642.74	639.20
9	769.22	770.81	774.28	805.51	768.23	766.31	776.65	772.65	775.87	770.74
10	812.01	814.07	818.33	854.03	811.14	809.55	819.98	815.09	818.88	814.05

Table 2. First ten natural frequencies of an elliptic paraboloid of arbitrary shape employing higher-order theories. Effect of the kinematic model on the accuracy of the solution.

f [Hz]	3D FEM	FSDT	TSDT	EDZ1	ED2	EDZ2	ED3	EDZ3	ED4	EDZ4
DOFs	720756	5046	10092	7569	7569	10092	10092	12615	12615	15138
1	68.95	69.40	69.36	71.17	69.30	69.22	69.53	69.36	69.47	69.32
2	142.37	143.03	143.26	145.98	142.40	142.21	143.67	142.89	143.49	142.86
3	321.60	325.31	325.48	330.90	322.59	321.99	325.66	323.59	325.01	323.49
4	395.48	397.32	398.09	410.07	396.61	396.02	399.60	397.63	398.97	397.60
5	486.92	490.53	490.64	502.73	488.19	487.44	491.76	489.38	490.97	489.10
6	644.87	649.16	649.58	665.85	645.89	644.92	650.70	647.47	649.72	646.67
7	811.44	820.12	821.09	847.21	812.22	810.61	820.89	815.87	819.34	814.71
8	852.97	859.66	862.08	882.93	854.22	852.19	863.90	857.32	862.00	856.91
9	989.03	995.68	998.63	1034.67	987.69	985.59	999.17	992.69	997.29	991.41
10	1062.61	1066.56	1066.40	1080.32	1064.68	1063.81	1067.82	1065.37	1066.88	1064.84

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